DISCRETE MATHEMATICS

Quantum double Schubert polynomials, quantum Schubert polynomials and Vafa–Intriligator formula

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Abstract

We study algebraic aspects of equivariant quantum cohomology algebra of the flag manifold. We introduce and study the quantum double Schubert polynomials $\tilde{\mathfrak{S}}_w(x,y)$, which are the Lascoux–Schützenberger type representatives of the equivariant quantum cohomology classes. Our approach is based on the quantum Cauchy identity. We define also quantum Schubert polynomials $\tilde{\mathfrak{S}}_w(x)$ as the Gram–Schmidt orthogonalization of some set of monomials with respect to the scalar product, defined by the Grothendieck residue. Using quantum Cauchy identity, we prove that $\tilde{\mathfrak{S}}_w(x)=\tilde{\mathfrak{S}}_w(x,y)|_{y=0}$ and as a corollary obtain a simple formula for the quantum Schubert polynomials $\tilde{\mathfrak{S}}_w(x)=\partial^{(y)}_{ww_0}\tilde{\mathfrak{S}}_{w_0}(x,y)|_{y=0}$. We also prove the higher genus analog of Vafa–Intriligator's formula for the flag manifolds and study the quantum residues generating function. We introduce the Ehresmann–Bruhat graph on the symmetric group and formulate the equivariant quantum Pieri rule. © 2000 Elsevier Science B.V. All rights reserved.

Résumé

Nous étudions les aspects algébriques de la cohomologie quantique de la varieété de drapeaux. Nous introduisons et étudions les polynômes de Schubert doubles quantiques $\tilde{\mathfrak{S}}_w(x,y)$, qui sont les représentants des classes de cohomologie équivariantes du type des polynômes de Lascoux-Schützenberger. Notre approche est fondée sur une identité de cauchy quantique. Nous définissons aussi les polynômes de Schubert quantiques par un procédé d'orthogonalisation de Gram–Schmidt, par rapport à un produit scalaire défini à l'aide d'un résidu de Grothendieck. Utilisant la formule de Cauchy quantique, nous montrons que $\tilde{\mathfrak{S}}_w(x)=\tilde{\mathfrak{S}}_w(x,y)|_{y=0}$, et comme corollaire, nous obtenons une formule simple pour les polynômes de Schubert quantiqes: $\tilde{\mathfrak{S}}_w(x)=\partial^{(y)}_{ww_0}\tilde{\mathfrak{S}}_{w_0}(x,y)|_{y=0}$. Nous prouvons aussi l'analogue, en genre plus élevé, de la formule de Vafa–Intriligator pour les variétés de drapeaux, et étudions la fonction génératrice des résidus quantiques. Nous introduisons enfin le graphe d'Ehresmann–Bruhat sur le groupe symétrique et énoncons la régle de Pieri quantique équivariante. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Double quantum Schubert polynomials; Ehresman-Bruhat graph; Quantum Pieri's rule

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1. Introduction

The cohomology ring of the flag variety $\mathrm{Fl}_n = \mathrm{SL}_n/B$ is isomorphic to the quotient ring of the polynomial ring by the ideal generated by symmetric polynomials without constant term. The Schubert cycles give a linear basis of the cohomology ring and they are represented by Schubert polynomials. Our aim is to introduce the notion of quantum double Schubert polynomials, which represent the Schubert cycles in the equivariant quantum cohomology ring [10,13], and investigate their properties. We refer to [23,25] for definition and basic properties of the quantum cohomology.

Our approach is based on the quantum Cauchy identity [17, Theorem 3]

$$\sum_{w \in S_n} \tilde{\mathfrak{S}}_w(x)\mathfrak{S}_{ww_0}(y) = \tilde{\mathfrak{S}}_{w_0}(x,y) := \prod_{k=1}^{n-1} \Delta_k(y_{n-k} \,|\, x_1,\ldots,x_k),$$

and on the Lascoux–Schützenberger type formula for the quantum double Schubert polynomials [17, Definition 4]

$$\tilde{\mathfrak{S}}_w(x,y) = \partial^{(y)}_{ww_0} \tilde{\mathfrak{S}}_{w_0}(x,y).$$

Let us explain briefly the content of our paper.

Following Givental and Kim [10], and Ciocan–Fontanine [4], we define the quantum elementary polynomials $\tilde{e}_1 := e_1(x\,|\,q),\ldots,\tilde{e}_n := e_n(x\,|\,q)$ by the formula

$$\det\begin{pmatrix} x_1+t & q_1 & 0 & \ldots & \ldots & \ldots & 0 \\ -1 & x_2+t & q_2 & 0 & \ldots & \ldots & 0 \\ 0 & -1 & x_3+t & q_3 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & -1 & x_{n-2}+t & q_{n-2} & 0 \\ 0 & \ldots & \ldots & 0 & -1 & x_{n-1}+t & q_{n-1} \\ 0 & \ldots & \ldots & \ldots & 0 & -1 & x_n+t \end{pmatrix}$$
$$= t^n + \tilde{e}_1 t^{n-1} + \tilde{e}_2 t^{n-2} + \cdots + \tilde{e}_n.$$

The defining ideal $\tilde{I}$ of the small quantum cohomology ring is generated by the quantum elementary polynomials, namely

$$QH^*(\mathrm{Fl}_n,\mathbf{Z}) := QH^*(\mathrm{Fl}_n) = \mathbf{Z}[x_1,\ldots,x_n;q_1,\ldots,q_{n-1}]/(\tilde{e}_1,\ldots,\tilde{e}_n). \tag{1}$$

In the classical case $q_1 = \cdots = q_{n-1} = 0$, on the quotient ring

$$\mathfrak{A} := \mathbf{Z}[x_1,\ldots,x_n]/(e_1(x),\ldots,e_n(x)) \simeq H^*(\mathrm{Fl}_n,\mathbf{Z})$$

exists a natural pairing $\langle f,g\rangle = \eta(\partial_{w_0}(fg))$ which comes from the intersection pairing in the homology group $H_*(\mathrm{Fl}_n,\mathbf{Z})$ of the flag variety. We can interpret the pairing $\langle , \rangle$ as the Grothendieck residue pairing with respect to the ideal I (see Section 2.5):

$$\langle f,g\rangle = \mathrm{Res}_I(fg) \quad \text{where } I = \tilde{I}|_{q=0}.$$

Our first observation, cf. [10, Theorem 3], is that a natural residue pairing (we call it the quantum residue pairing)

$$\langle f, g\rangle_Q = \mathrm{Res}_{\tilde{I}}(fg)$$

on the quotient ring $\tilde{\mathfrak{A}}:=\mathbf{Z}[x_1,\ldots,x_n]/\tilde{I}$ (see [11, Chapter 5], for definition of the residue pairing) corresponds to the intersection pairing in quantum cohomology $QH^*(\mathrm{Fl}_n,\mathbf{Z})$ under a natural isomorphism (1).

It is well-known (see, e.g. [20–22]) that the classical Schubert polynomials form an orthonormal basis with respect to the pairing $\langle,\rangle$ in the cohomology ring of flag manifold, and also give a linear basis in the quantum cohomology ring $QH^*(\mathrm{Fl}_n,\mathbf{Z})$ [10,23]. However, the classical Schubert polynomials do not orthogonal with respect to the quantum pairing any more. Thus, it is natural to ask: what kind of polynomials can one obtain applying the Gram–Schmidt orthogonalization to the classical Schubert polynomials with respect to the quantum pairing $\langle,\rangle_Q$? Omitting some details with ordering (see Definition 5), the answer is: quantum Schubert polynomials.

Our second observation is: to work with the equivariant quantum cohomology algebra [10,13] is more convenient than with quantum cohomology ring itself. The main reason is that one can find the Lascoux–Schützenberger type representative for any equivariant quantum cohomology class. In other words, each quantum double Schubert polynomial $\tilde{\mathfrak{S}}_w(x,y)$ can be obtained from the top one by using the divided difference operators acting on the y variables.

Proposition–Definition A. *Let $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_n)$ be two sets of variables, and*

$$\tilde{\mathfrak{S}}^{(q)}_{w_0}(x,y):=\prod_{i=1}^{n-1}\Delta_i(y_{n-i}\,|\,x_1,\ldots,x_i),$$

where $\Delta_k(t\,|\,x_1,\ldots,x_k):=\sum_{j=0}^{k}t^{k-j}e_j(x_1,\ldots,x_k\,|\,q_1,\ldots,q_{k-1})$ is the generating function for the quantum symmetric polynomials in $x_1,\ldots,x_k$.

Then $\tilde{\mathfrak{S}}^{(q)}_{w}(x,y)=\partial^{(y)}_{ww_0}\tilde{\mathfrak{S}}^{(q)}_{w_0}(x,y)$.

Proof follows from a description of the equivariant cohomology of flag variety given by Givental and Kim, see [10], Section 3; [13].

We define the quantum Schubert polynomials $\tilde{\mathfrak{S}}^{(q)}_{w}(x)$ as Gram–Schmidt's orthogonalization of the set of lexicographically ordered monomials $\{x^I\,|\,I\subset(n-1,n-2,\ldots,1,0)\}$ with respect to the quantum pairing $\langle,\rangle_Q$, see Definition 5. One of our main results is the quantum analog of Cauchy's identity for (classical) Schubert polynomials [21,22, Eq. (5.10)].

Theorem B (Quantum Cauchy's identity).

$$\sum_{w\in S_n}\tilde{\mathfrak{S}}^{(q)}_{w}(x)\mathfrak{S}_{ww_0}(y)=\tilde{\mathfrak{S}}^{(q)}_{w_0}(x,y). \tag{2}$$

We give a geometric proof of Theorem B in Section 7 using the arguments due to Ciocan-Fontanine [4]; more particularly, we reduce directly a proof of Theorem B to that of the following geometric statement:

Lemma. *Let $I \subset \delta = (n-1, n-2, \ldots, 1, 0)$ and $w \in S_n$ be a permutation, then*

$$\langle \tilde{e}_I(x), \tilde{\mathfrak{S}}_w(x) \rangle_Q = \langle e_I(x), \mathfrak{S}_w(x) \rangle, \tag{3}$$

where $e_I(x) := \prod_{k=1}^{n-1} e_{i_k}(x_1, \ldots, x_{n-k})$ (resp. $\tilde{e}_I(x) := \prod_{k=1}^{n-1} \tilde{e}_{i_k}(x_1, \ldots, x_{n-k} \mid q_1, \ldots, q_{n-k-1})$) *is the elementary polynomial* (*resp. quantized elementary polynomial*), *see Section* 5.2.

It is formula (3) that we prove in Section 7 using geometric arguments from [4,13]. As a product, it follows from our proof that the quantum Schubert polynomials $\hat{\mathfrak{S}}_w(x)$ defined geometrically (see [4], and Section 6) coincide with those defined algebraically (see Definition 5):

$$\hat{\mathfrak{S}}_w(x) \equiv \tilde{\mathfrak{S}}_{w^{-1}}(x) \, (\mathrm{mod}\, \tilde{I}).$$

It is interesting to note (see Section 5, (20)) that the intersection numbers $\langle e_I(x), \mathfrak{S}_w(x) \rangle$ (which are nonnegative!) are precisely the coefficients of corresponding Schubert polynomial:

$$\mathfrak{S}_w(x) = \sum_{I \subset \delta} \langle e_I(x), \mathfrak{S}_w(x) \rangle x^{\delta - I}.$$

The quantum Cauchy formula (2) plays an important role in our approach to the quantum Schubert polynomials. As a direct consequence of (2), we obtain the Lascoux–Schützenberger-type formula for quantum Schubert polynomials (cf. Proposition–Definition A).

Theorem C. *Let $\tilde{\mathfrak{S}}_{w_0}(x, y)$ be as in Proposition–Definition A, then*

$$\tilde{\mathfrak{S}}_w(x) = \partial^{(y)}_{ww_0} \tilde{\mathfrak{S}}_{w_0}(x, y)|_{y=0}.$$

In Section 5 we introduce a quantization map $P_n \to \bar{P}_n, \; f \mapsto \tilde{f}$. The quantization is a linear map which preserves the pairings, i.e.,

$$\langle \tilde{f}, \tilde{g} \rangle_Q = \langle f, g \rangle, \quad f, g \in P_n.$$

Using the quantum Cauchy formula (2), we prove that quantum double Schubert polynomials are the quantization of classical ones. Another class of polynomials having a good quantization is the set of elementary polynomials

$$e_I(x) := \prod_{k=1}^{n-1} e_{i_k}(x_1, \ldots x_{n-k}), \qquad I = (i_1, \ldots, i_{n-1}) \subset \delta.$$

It follows from Theorem B that quantization $\tilde{e}_I(x)$ of elementary polynomial $e_I(x)$ is given by

$$\tilde{e}_I(x) = \prod_{k=1}^{n-1} e_{i_k}(x_1,\ldots,x_{n-k} \mid q_1,\ldots,q_{n-k-1}).$$

In Section 5.2 we consider a problem of how to quantize monomials. We prove the following relations for quantum monomials (see also Remark 7):

$$\tilde{x}^I = \sum_{w \in S_n} \eta(\partial_w x^I)\tilde{\mathfrak{S}}_w(x), \quad I \subset \delta,$$

$$\tilde{\mathfrak{S}}_{w_0}(x,y) = \sum_{I \subset \delta} \tilde{x}^I e_{\delta - I}(y).$$

In Section 8.1 we give a proof of the higher genus analog of the Vafa–Intriligator type formula for the flag manifold.

In Section 8.3 we study a problem of how to compute the quantum residues. This is important for computation of the (small quantum cohomology ring) correlation functions and the Gromov–Witten invariants. We introduce the generating function

$$\Psi(t) = \left\langle \prod_{i=1}^{n-1} \frac{t_i}{t_i - x_i} \right\rangle$$

for quantum residues and give a characterization of this function as the unique solution to some system of differential equations, see Proposition 14.

In Section 9 we introduce the Ehresmann–Bruhat graph and give a sketch of a proof of the equivariant quantum Pieri rule. Details will appear elsewhere.

We would like to mention that in a recent paper 'Quantum Schubert polynomials', Fomin et al. [7] developed a different approach to the theory-of quantum Schubert polynomials, based on the remarkable family of commuting operators X_i [7, Eq. (3.2)]. Among main results, obtained by Fomin et al. are definitions, orthogonality, quantum Monk's formula and other properties of quantum Schubert polynomials; definition of quantization map and quantum multiplication.

Although some overlap with the paper of Fomin et al. occurs, our works were done independently were based on the different approaches which allowed as to obtain mutually complementary results.

2. Classical schubert polynomials

In this section we give a brief review of the theory of Schubert polynomials created by Lascoux and Schützenberger. In exposition we follow to the Macdonald's book [21] where proofs and details can be found.

2.1. Divided differences

Let $x_1,\dots,x_n,\dots$ be independent variables, and let

$$P_n{:=}\mathbf{Z}[x_1,\dots,x_n] \text{ for } n\geqslant 1 \quad \text{and} \quad P_\infty{:=}\mathbf{Z}[x_1,x_2,\dots]=\bigcup_{n=1}^{\infty} P_n. \tag{4}$$

Let us denote by $\Lambda_n{:=}\mathbf{Z}[x_1,\dots,x_n]^{S_n}\subset P_n$ the ring of symmetric polynomials in $x_1,\dots,x_n$, and by $H_n{:=}\{\sum_{I=(i_1,\dots,i_n)} a_I x^I \mid a_I\in\mathbf{Z},\ 0\leqslant i_k\leqslant n-k, \forall k\}$ the additive subgroup of P_n spanned by all monomials $x^I{:=}x_1^{i_1}x_2^{i_2}\dots x_n^{i_n}$ with $I\subset\delta{:=}\delta_n=(n-1,n-2,\dots,1,0)$. For $1\leqslant i\leqslant n-1$ let us define a linear operator ∂_i acting on P_n

$$(\partial_i f)(x)=\frac{f(x_1,\dots,x_i,x_{i+1},\dots,x_n)-f(x_1,\dots,x_{i+1},x_i,\dots,x_n)}{x_i-x_{i+1}}. \tag{5}$$

Divided difference operators ∂_i satisfy the following relations:

$$\begin{aligned}
&\partial_i^2=0,\\
&\partial_i\partial_j=\partial_j\partial_i \quad \text{if } |i-j|>1,\\
&\partial_i\partial_{i+1}\partial_i=\partial_{i+1}\partial_i\partial_{i+1},
\end{aligned} \tag{6}$$

and the Leibnitz rule

$$\partial_i(fg)=\partial_i(f)g+s_i(f)\partial_i(g). \tag{7}$$

It follows from (7) that ∂_i is a Λ_n-linear operator.

For any permutation $w\in S_n$, let us denote by $R(w)$ the set of reduced words for w, i.e. sequences $(a_1,\dots,a_p)$ such that $w=s_{a_1}\cdots s_{a_p}$, where $p=l(w)$ is the length of permutation $w\in S_n$, and $s_i=(i,i+1)$ is the simple transposition that interchanges i and $i+1$.

For any sequence $\boldsymbol{a}=(a_1,\dots,a_p)$ of positive integers, we define

$$\partial_{\boldsymbol{a}}=\partial_{a_1}\cdots\partial_{a_p}.$$

Proposition 1 (Macdonald [21, Eqs. (2.5) and (2.6)]).
- *If* $\boldsymbol{a},\boldsymbol{b}\in R(w)$, *then* $\partial_{\boldsymbol{a}}=\partial_{\boldsymbol{b}}$.
- *If* $\boldsymbol{a}$ *is not reduced, then* $\partial_{\boldsymbol{a}}=0$.

From Proposition 1 it follows that an operator

$$\partial_w=\partial_{\boldsymbol{a}}$$

is well-defined, where $\boldsymbol{a}$ is any reduced word for w. By (7), the operators ∂_w, $w\in S_n$, are Λ_n linear, i.e., if $f\in\Lambda_n$, then

$$\partial_w(fg)=f\partial_w(g).$$

2.2. Schubert polynomials

Let $\delta{:=}\delta_n=(n-1,n-2,\dots,1,0)$, so that $x^\delta=x_1^{n-1}x_2^{n-2}\dots x_{n-1}$.

Definition 1 (*Lascoux–Schützenberger* [19]). For each permutation $w \in S_n$ the Schubert polynomial $\mathfrak{S}_w$ is defined to be

$$\mathfrak{S}_w(x) = \partial_{w^{-1}w_0}(x^\delta),$$

where w_0 is the longest element of S_n.

Proposition 2 (Macdonald [21, Eqs. (4.2), (4.5), (4.11) and (4.15)]).

- *Let $v, w \in S_n$. Then*

$$\partial_v \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{wv^{-1}} & \text{if } l(wv^{-1}) = l(w) - l(v), \\ 0 & \text{otherwise.} \end{cases}$$

- (*Stability*) *Let $m > n$ and let $i : S_n \hookrightarrow S_m$ be the natural embedding. Then*

$$\mathfrak{S}_w = \mathfrak{S}_{i(w)}.$$

- *The Schubert polynomials $\mathfrak{S}_w$, $w \in S_n$ form a* **Z***-basis of H_n.*
- (*Monk's formula*) *Let $f = \sum_{i=1}^n \alpha_i x_i$, $w \in S_n$. Then*

$$f\mathfrak{S}_w = \sum (\alpha_i - \alpha_j)\mathfrak{S}_{wt_{ij}},$$

$$\partial_w(fg) = w(f)\partial_w g + \sum (\alpha i - \alpha j)\partial_{wt_{ij}} g,$$

where t_{ij} is the transposition that interchanges i and j, and both sums are over all pairs $i < j$ such that $l(wt_{ij}) = l(w) + 1$.

2.3. Scalar product

Let us define a scalar product on P_n with values in Λ_n, by the rule

$$\langle f, g \rangle = \partial_{w_0}(fg), \quad f, g \in P_n, \tag{8}$$

where w_0 is the longest element of S_n. The scalar product $\langle , \rangle$ defines a non-degenerate pairing $\langle , \rangle_0$ on the quotient ring $P_n/I_n \cong H^*(\mathrm{Fl}_n, \mathbf{Z})$, where I_n is the ideal in P_n generated by the elementary symmetric polynomials $e_1(x), \ldots, e_n(x)$.

Proposition 3 (Macdonald [21, Eqs. (5.3), (5.4), (5.6), (4.13) and (5.10)]).

- *If $f \in \Lambda_n$, then $\langle fh, g \rangle = f\langle h, g \rangle$.*
- *If $f, g \in P_n$, $w \in S_n$, then $\langle \partial_w f, g \rangle = \langle f, \partial_{w^{-1}} g \rangle$.*
- (*Orthogonality*) *If $l(u) + l(v) = l(w_0)$, then*

$$\langle \mathfrak{S}_u, \mathfrak{S}_v \rangle = \begin{cases} 1 & \text{if } u = w_0 v, \\ 0 & \text{otherwise.} \end{cases}$$

- *The Schubert polynomials $\mathfrak{S}_w$, $w \in S_n$, form a Λ_n-basis of P_n.*
- *The Schubert polynomials $\mathfrak{S}_w$, $w \in S^{(n)}$, form a* **Z***-basis of P_n, where for each $n \geqslant 1$, $S^{(n)}$ is the set of all permutations w such that the code w has length $\leqslant n$.*

• (*Cauchy's formula*)

$$\sum_{w \in S_n} \mathfrak{S}_w(x)\mathfrak{S}_{ww_0}(y) = \prod_{i+j \leqslant n} (x_i + y_j).$$

Proposition 4. *Schubert polynomials are uniquely characterized by the following properties*:

1. (*Orthogonality*)

$$\langle \mathfrak{S}_u, \mathfrak{S}_v \rangle_0 = \begin{cases} 1 & if\ u = w_0 v, \\ 0 & otherwise. \end{cases}$$

2. *Let w be a permutation in S_n and $c(w) = (c_1, c_2, \ldots, c_n)$ its code, then*

$$\mathfrak{S}_w(x) = x^{c(w)} + \sum \alpha_I x^I,$$

where $I \subset \delta$, $\alpha_I > 0$ and I lexicographically smaller than $c(w)$.

Remark 1. (1) (Definition of the code [21, p. 9]). For a permutation $w \in S_n$, we define

$$c_i = \sharp\{j \mid i < j, w(i) > w(j)\}.$$

The sequence $c(w) = (c_1, c_2, \ldots, c_n)$ is called the code of w.

(2) Schubert polynomials are obtained as Gram–Schmidt's orthogonalization of the set of monomials $\{x^I\}_{I \subset \delta}$ ordered lexicographically.

2.4. Double Schubert polynomials

Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ be two sets of independent variables, and

$$\mathfrak{S}_{w_0}(x, y) := \prod_{i+j \leqslant n} (x_i + y_j).$$

Definition 2 (*Lascoux–Schützenberger* [20]). For each permutation $w \in S_n$, the double Schubert polynomial $\mathfrak{S}_w(x, y)$ is defined to be

$$\mathfrak{S}_w(x, y) = \partial^{(x)}_{w^{-1}w_0} \mathfrak{S}_{w_0}(x, y),$$

where divided difference operator $\partial^{(x)}_{w^{-1}w_0}$ acts on the x variables.

Proposition 5 (Macdonald [21, Eqs. (6.3) and (6.8)]).

• $\mathfrak{S}_w(x, y) = \sum_u \mathfrak{S}_u(x)\mathfrak{S}_{uw^{-1}}(y)$, *summed over all $u \in S_n$, such that $l(u) + l(uw^{-1}) = l(w)$;*

• (*Interpolation formula*) *For all $f \in \mathbf{Z}[x_1, \ldots, x_n]$ we have*

$$f(x) = \sum_w \mathfrak{S}_w(x, -y)\partial^{(y)}_w f(y)$$

summed over all permutations $w \in S^{(n)}$.

- *Double Schubert polynomials appear in algebra and geometry as cohomology classes related to degeneracy loci of flagged vector bundles, see, e.g.*, [9]. *If* $h: E \to F$ *is a map of rank n vector bundles on a smooth variety* X,

  $$E_1 \subset E_2 \subset \cdots \subset E_n = E, \ F := F_n \to F_{n-1} \to \cdots \to F_1$$

  are flags of subbundles and quotient bundles, then there is a degeneracy locus $\Omega_w(h)$ *for each permutation* w *in the symmetric group* S_n, *described by the conditions*

  $$\Omega_w(h) = \{x \in X \mid \mathrm{rank}\ (E_p(x) \to F_q(x)) \leqslant \#\{i \leqslant q, \ w_i \leqslant p\}, \forall p, q\}.$$

  For generic h, $\Omega_w(h)$ *is irreducible, codim* $\Omega_w(h) = l(w)$, and the class $[\Omega_w(h)]$ *of this locus in the Chow ring of* X *is equal to the double Schubert polynomial* $\mathfrak{S}_{w_0 w}(x, -y)$, *where*

  $$x_i = c_1(\ker(F_i \to F_{i-1})),$$

  $$y_i = c_1(E_i/E_{i-1}), \ 1 \leqslant i \leqslant n.$$

  It is well-known [9] *that the Chow ring of flag variety* Fl_n *admits the following description*:

  $$\mathrm{CH}^*(\mathrm{Fl}_n) \cong \mathbf{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n]/J,$$

  where J *is the ideal generated by*

  $$e_i(x_1, \ldots, x_n) - e_i(y_1, \ldots, y_n), \quad 1 \leqslant i \leqslant n,$$

  and $e_i(x)$ *is the ith elementary symmetric function in the variables* $x_1, \ldots, x_n$.
- (*Lascoux and Schützenberger* [20] *and Kohnert and Veigneau* [18]). *The ring* $\mathbf{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n]/J$ *is a free module of dimension* $n!$ *over the ring* R, *and its basis is either* $\mathfrak{S}_w(x)$, *or* $\mathfrak{S}_w(x, y)$, $w \in S_n$, *where*

  $$R := \frac{\mathbf{Z}[x_1, \ldots, x_n] \otimes \mathbf{Sym}[y_1, \ldots, y_n]}{J}.$$

2.5. Residue pairing

Let I be an ideal in $\bar{P}_n = R[x_1, \ldots, x_n]$, $R \subset \boldsymbol{C}$, generated by a regular system of parameters $\varphi_1, \ldots, \varphi_n$, and $\mathfrak{A} := \bar{P}_n/I$.

Proposition 6 (Griffith and Harris [11] and Eisenbud and Levine [6]).
- $\dim_R \mathfrak{A} < \infty$.
- $\mathscr{H} := \det(\partial\varphi_i/\partial x_j) \notin I$.

Let $d_0 := \deg \mathscr{H}$, where we assume that $\deg x_i = 1$ for all $1 \leqslant i \leqslant n$.

Proposition 7 (Eisenbud and Levine [6]).
- *If* $f \in \bar{P}_n$ *and* $\deg f = d_0$, *then there exists a nonzero* $\alpha \in R$ *such that* $f \equiv (\alpha/n!)\mathscr{H}$ $(\mathrm{mod}\, I)$.

- *If* $f \in \bar{P}_n$, $f \neq 0$, *and* $\deg f > d_0$, *then there exists* $g \in \bar{P}_n$ *such that* $\deg g \leqslant d_0$ *and* $g \equiv f (\operatorname{mod} I)$.

Definition 3 (*Grothendieck residue with respect to the ideal* I). Let $f \in \bar{P}_n$ and $\deg f < d_0$, then we define $\operatorname{Res}_I(f) = 0$. If $\deg f = d_0$, then $f \equiv (\alpha/n!)\mathscr{H} \ (\operatorname{mod} I)$ and we define $\operatorname{Res}_I(f) := \alpha$. Finally, if $\deg f > d_0$, then choose $g \in \bar{P}_n$ such that $g \equiv f \ (\operatorname{mod} I)$ and $\deg g \leqslant d_0$, and define

$$\operatorname{Res}_I(f) := \operatorname{Res}_I(g).$$

We will use also notation $\langle f \rangle_I$ instead of $\operatorname{Res}_I(f)$. Finally, let us define a residue pairing $\langle , \rangle_I$ on $\bar{P}_n$ using the Grothendieck residue

$$\langle f, g \rangle_I = \operatorname{Res}_I(f, g), \quad f, g \in \bar{P}_n.$$

Proposition 8 (Griffiths and Harris [11]).
- *If* $f \in I$, *then* $\operatorname{Res}_I(f) = 0$.
- *The residue pairing* $\langle , \rangle_I$ *induces a non-degenerate pairing on* $\mathfrak{A} = \bar{P}/I$.

We will use this general construction of residue pairing in the following two cases:

(i) $R = \mathbf{Z}$, $I_n \subset P_n$ is an ideal generated by elementary symmetric polynomials $e_1(x), \ldots, e_n(x)$. It is well-known that if $\mathrm{Fl}_n := \mathrm{SL}(n)/B$ is the flag variety of type A_{n-1}, then

$$H^*(\mathrm{Fl}_n, \mathbf{Z}) \simeq P_n/I_n,$$

and residue pairing $\langle , \rangle$ on P_n/I_n coincides with the scalar product on P_n/I_n induced by (8).

(ii) $R = \mathbf{Z}[q_1, \ldots, q_{n-1}]$, $\tilde{I}_n \subset \bar{P}_n$ is an ideal generated by the quantum elementary symmetric functions $\tilde{e}_1(x), \ldots, \tilde{e}_n(x)$. It is a result of Givental and Kim, and Ciocan–Fontanine that

$$QH^*(\mathrm{Fl}_n) \simeq \bar{P}_n/\tilde{I}_n,$$

and the residue pairing defined by $\tilde{I}_n$ may be naturally identified with the intersection form on the quantum cohomology ring. We will call this residue pairing as quantum pairing on $\bar{P}_n/\tilde{I}_n$ and denote it by $\langle , \rangle_Q$.

3. Quantum double Schubert polynomials

Quantum double Schubert polynomials are closely related to the equivariant quantum cohomology. Let us remind the result of Givental and Kim [10] (see also [13]) on the structure of the equivariant quantum cohomology algebra of the flag variety Fl_n:

$$QH^*_{U_n}(\mathrm{Fl}_n) \cong \mathbf{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n, q_1, \ldots, q_{n-1}]/\tilde{J},$$

where the ideal $\tilde{J}$ is generated by

$$e_i(x_1, \ldots, x_n \mid q_1, \ldots, q_{n-1}) - e_i(y_1, \ldots, y_n), \quad 1 \leqslant i \leqslant n.$$

In classical case $q=0$, the double Schubert polynomials $\mathfrak{S}_w(x,y)$ represent the equivariant cohomology classes [9]. Quantum double Schubert polynomials have to play a similar role for the quantum equivariant cohomology ring. First let us define the 'top' quantum double Schubert polynomial $\tilde{\mathfrak{S}}_{w_0}(x,y)$.

Let $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)$ be two sets of variables, put

$$\tilde{\mathfrak{S}}_{w_0}(x,y):=\tilde{\mathfrak{S}}_{w_0}^{(q)}(x,y)=\prod_{i=1}^{n-1}\Delta_i(y_{n-i}\,|\,x_1,\dots,x_i),$$

where $\Delta_k(t\,|\,x_1,\dots,x_k):=\sum_{j=0}^{k}t^{k-j}e_j(x_1,\dots,x_k\,|\,q_1,\dots,q_{k-1})$ is the generating function for the quantum elementary polynomials in $x_1,\dots,x_k$, i.e.

$$\Delta_k(t|x):=\sum_{i=0}^{k}e_i(x|q)t^{k-i}$$

$$=\det\begin{pmatrix} x_1+t & q_1 & 0 & \dots & \dots & \dots & 0\\ -1 & x_2+t & q_2 & 0 & \dots & \dots & 0\\ 0 & -1 & x_3+t & q_3 & 0 & \dots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & \dots & 0 & -1 & x_{k-2}+t & q_{k-2} & 0\\ 0 & \dots & \dots & 0 & -1 & x_{k-1}+t & q_{k-1}\\ 0 & \dots & \dots & \dots & 0 & -1 & x_k+t \end{pmatrix} \tag{9}$$

Definition 4. For each permutation $w\in S_n$, the quantum double Schubert polynomial $\tilde{\mathfrak{S}}_w(x,y)$ is defined to be

$$\tilde{\mathfrak{S}}_w(x,y)=\partial_{ww_0}^{(y)}\tilde{\mathfrak{S}}_{w_0}(x,y),$$

where divided difference operator $\partial_{ww_0}^{(y)}$ acts on the y variables.

Remark 2. (i) In the 'classical limit' $q_1=\cdots=q_{n-1}=0$,

$$\tilde{\mathfrak{S}}_w(x,y)|_{q=0}=\partial_{ww_0}^{(y)}\mathfrak{S}_{w_0}(x,y)=\mathfrak{S}_{w^{-1}}(y,x)=\mathfrak{S}_w(x,y),$$

i.e. $\tilde{\mathfrak{S}}_w(x,y)|_{q=0}=\mathfrak{S}_w(x,y)$.

(ii) (*Stability*) Let $m>n$ and let $i:S_n\hookrightarrow S_m$ be the embedding. Then

$$\tilde{\mathfrak{S}}_w(x,y)=\tilde{\mathfrak{S}}_{i(w)}(x,y).$$

(iii) One can check that the ring $\mathbf{Z}[x_1,\dots,x_n,y_1,\dots,y_n,q_1,\dots,q_{n-1}]/\tilde{J}$ is a free module of dimension $n!$ over the quotient ring $\tilde{R}$ with basis either $\tilde{\mathfrak{S}}_w(x)$ or $\tilde{\mathfrak{S}}_w(x,y)$, $w\in S_n$, where

$$\tilde{R}:=\frac{\mathbf{Z}[x_1,\dots,x_n,q_1,\dots,q_{n-1}]\otimes\mathbf{Sym}[y_1,\dots,y_n]}{\tilde{J}}.$$

Example. Quantum double Schubert polynomials for S_3:

$$\tilde{\mathfrak{S}}_{s_1s_2s_1}(x,y)=(x_1+y_2)(x_1+y_1)(x_2+y_1)+q_1(x_1+y_2),$$
$$\tilde{\mathfrak{S}}_{s_2s_1}(x,y)=(x_1+y_1)(x_1+y_2)-q_1,$$
$$\tilde{\mathfrak{S}}_{s_1s_2}(x,y)=(x_1+y_1)(x_2+y_1)+q_1,$$
$$\tilde{\mathfrak{S}}_{s_1}(x,y)=x_1+y_1,$$
$$\tilde{\mathfrak{S}}_{s_2}(x,y)=x_1+x_2+y_1+y_2,$$
$$\tilde{\mathfrak{S}}_{\mathrm{id}}(x,y)=1.$$

Theorem 1. *Let $z=(z_1,\ldots,z_n)$ be a third set of variables. Then*

$$\langle\tilde{\mathfrak{S}}_{w_0}(x,y),\tilde{\mathfrak{S}}_{w_0}(x,z)\rangle_Q^{(x)}=C(y,z), \tag{10}$$

where the upper index x means that the quantum pairing is taken in the x variables, and

$$C(x,y)=\sum_{w\in S_n}\mathfrak{S}_w(x)\mathfrak{S}_{w_0w}(y)$$

is the 'canonical' element in the tensor product $H^*(\mathrm{Fl}_n)\otimes H^*(\mathrm{Fl}_n)$.

Theorem 1 plays an important role in our approach to the quantum Schubert polynomials. We will give the proof later, but now let us consider some applications of formula (10).

4. Quantum Schubert polynomials

4.1. Definition

Let us recall the result of Givental and Kim [10] and Ciocan-Fontanine [5] on the structure of the small quantum cohomology ring of flag variety Fl_n

$$QH^*(\mathrm{Fl}_n)\cong\mathbf{Z}[x_1,\ldots,x_n,q_1,\ldots,q_{n-1}]/\tilde{I},$$

where ideal $\tilde{I}$ is generated by the quantum elementary polynomials

$$\tilde{e}_i(x):=e_i(x_1,\ldots,x_n|q_1,\ldots,q_{n-1}),\quad 1\leqslant i\leqslant n$$

with generating function $\Delta_n(t|x)$, see (9).

We define a pairing on the ring of polynomials $\mathbf{Z}[x;q]$ and the quantum cohomology ring $QH^*(\mathrm{Fl}_n)\simeq\mathbf{Z}[x;q]/\tilde{I}$ using the Grothendieck residue

$$\langle f,g\rangle_Q=\mathrm{Res}_{\tilde{I}}(fg),\ f,g\in\mathbf{Z}[x_1,\ldots,x_n,q_1,\ldots,q_{n-1}].$$

Then

(1) $\langle f,g\rangle_Q=0$ if $f\in\tilde{I}$;

(2) $\langle f,g\rangle_Q$ defines a nondegenerate pairing in $QH^*(\mathrm{Fl}_n)$.

Definition 5. Define the quantum Schubert polynomials $\tilde{\mathfrak{S}}_w := \tilde{\mathfrak{S}}_w(x)$ as Gram–Schmidt's orthogonalization of the set of lexicographically ordered monomials $\{x^I \mid I \subset \delta\}$ with respect to the quantum residue pairing $\langle f, g \rangle_Q$:

(1) $\langle \tilde{\mathfrak{S}}_u, \tilde{\mathfrak{S}}_v \rangle_Q = \langle \mathfrak{S}_u, \mathfrak{S}_v \rangle = \begin{cases} 1 & \text{if } v = w_0 u \\ 0 & \text{otherwise} \end{cases}$

(2) $\tilde{\mathfrak{S}}_w(x) = x^{c(w)} + \sum_{I < c(w)} a_I(q) x^I$, where $a_I(q) \in \mathbf{Z}[q_1, \ldots, q_{n-1}]$ and $I < c(w)$ means the lexicographic order.

Here $c(w)$ is the code of a permutation $w \in S_n$ [21, p. 9].

Remark 3. This definition is the analog of the characterization of Schubert polynomials from Proposition 4.

Example. For the symmetric group S_3, we have

$$\langle x_1^2 x_2, x_1^2 \rangle_Q = q_1, \quad \langle x_1^2 x_2, x_1 x_2 \rangle_Q = -2q_1.$$

Consequently,

$$\tilde{\mathfrak{S}}_1 = x_1, \quad \tilde{\mathfrak{S}}_2 = x_1 + x_2, \quad \tilde{\mathfrak{S}}_{12} = x_1 x_2 + q_1, \quad \tilde{\mathfrak{S}}_{21} = x_1^2 - q_1, \quad \tilde{\mathfrak{S}}_{121} = x_1^2 x_2 + q_1 x_1.$$

Let us remark that in our example ($n = 3$) $\tilde{\mathfrak{S}}_{121} = \tilde{\mathfrak{S}}_{w_0^{(3)}}(x) = \tilde{e}_1(x_1)\tilde{e}_2(x_1, x_2)$. More generally, it follows from Definition 4 that

$$\tilde{\mathfrak{S}}_{w_0}(x) = \tilde{e}_1(x_1)\tilde{e}_2(x_1, x_2) \cdots \tilde{e}_{n-1}(x_1, \ldots x_{n-1}).$$

In other words, the quantum Schubert polynomial corresponding to the longest element of the symmetric group S_n, is equal to the product of all principal minors of the Jacobi matrix $(\partial e_i(x|q)/\partial x_j)_{1 \leqslant i,j \leqslant n}$.

4.2. Orthogonality

We use the Jack–Macdonald type definition [22, Chapter VI] of the quantum Schubert polynomials, see Definition 5. On this way the orthogonality of quantum Schubert polynomials is valid by 'definition'. We are going to prove that the $y = 0$ specialization of quantum double Schubert polynomials $\tilde{\mathfrak{S}}_w(x, 0)$ also satisfies the conditions (1) and (2) of Definition 5. As a corollary, we obtain that the specialization $\tilde{\mathfrak{S}}_w(x, 0)$ coincides with the quantum Schubert polynomial $\tilde{\mathfrak{S}}_w(x)$ from Definition 5.

Theorem 2. *Let $v, w \in S_n$. Then*

$$\langle \tilde{\mathfrak{S}}_v(x, 0), \tilde{\mathfrak{S}}_w(x, 0) \rangle_Q = \begin{cases} 1 & \text{if } w = w_0 v, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let us apply the operator $\partial^{(y)}_{vw_0} \partial^{(z)}_{ww_0}$ to the both sides of (10). The LHS gives

$$\partial^{(y)}_{vw_0} \partial^{(z)}_{ww_0} \langle \tilde{\mathfrak{S}}_{w_0}(x, y), \tilde{\mathfrak{S}}_{w_0}(x, z) \rangle^{(x)}_Q = \langle \tilde{\mathfrak{S}}_v(x, y), \tilde{\mathfrak{S}}_w(x, z) \rangle^{(x)}_Q.$$

The RHS transforms to the following form $\sum_{u\in S_n}\partial^{(y)}_{vw_0}\mathfrak{S}_u(y)\partial^{(z)}_{ww_0}\mathfrak{S}_{w_0u}(z)$. Now taking $y=z=0$ we obtain an equality

$$\langle\tilde{\mathfrak{S}}_v(x,0)\tilde{\mathfrak{S}}_w(x,0)\rangle_Q=\sum_{u\in S_n}\eta(\partial_{vw_0}\mathfrak{S}_u)\eta(\partial_{ww_0}\mathfrak{S}_{w_0u}), \tag{11}$$

where $\eta: P_n\to\mathbf{Z}$ is the homomorphism defined by $\eta(x_i)=0$ $(1\leqslant i\leqslant n)$. It is clear that

$$\eta(\partial_v\mathfrak{S}_u)=\begin{cases}1 & \text{if } v=u,\\ 0 & \text{otherwise.}\end{cases}$$

Thus, the RHS of (11) is equal to 1 if $w_0ww_0=vw_0$ and is equal to 0 otherwise. □

Remark 4. Orthogonality of quantum Schubert polynomials was proven in [7], using a combinatorial definition [7, Section 5]; the proof is highly nontrivial.

4.3. Quantum Cauchy formula

Theorem 3. *Let* $\tilde{\mathfrak{S}}_w(x):=\tilde{\mathfrak{S}}_w(x,0,)$. *Then*

$$\sum_{w\in S_n}\tilde{\mathfrak{S}}_w(x)\mathfrak{S}_{ww_0}(y)=\tilde{\mathfrak{S}}_{w_0}(x,y). \tag{12}$$

Proof. Let us apply the divided difference operator $\partial^{(z)}_{ww_0}$ to the both sides of (10) and then take $z=0$. The right-hand side transforms to the following form:

$$\sum_{u\in S_n}\mathfrak{S}_u(y)\partial^{(z)}_{ww_0}\mathfrak{S}_{w_0u}(z)|_{z=0}=\mathfrak{S}_{w_0ww_0}(y).$$

As for the LHS, it takes the form $\langle\tilde{\mathfrak{S}}_{w_0}(x,y),\tilde{\mathfrak{S}}_w(x)\rangle_Q$. Hence,

$$\langle\tilde{\mathfrak{S}}_{w_0}(x,y),\tilde{\mathfrak{S}}_w(x)\rangle_Q=\mathfrak{S}_{w_0ww_0}(y).$$

The last identity is equivalent to (12). □

More generally, we have

Proposition 9.

$$\sum_{w\in S_n}\tilde{\mathfrak{S}}_w(x,z)\mathfrak{S}_{ww_0}(y,-z)=\tilde{\mathfrak{S}}_{w_0}(x,y), \tag{13}$$

$$\sum_{u\in S_n,\, l(u)+l(uw^{-1})=l(w)}\tilde{\mathfrak{S}}_u(x,z)\mathfrak{S}_{uw^{-1}}(y,-z)=\tilde{\mathfrak{S}}_w(x,y). \tag{14}$$

Proof. Let us apply the Interpolation formula to $f(x)=\tilde{\mathfrak{S}}_{w_0}(x,y)$ and then the divided difference operator $\partial^{(y)}_{ww_0}$. □

Corollary 1.

$$C^{(q,q')}(x,y):=\sum_{w\in S_n}\tilde{\mathfrak{S}}^{(q)}_w(x)\tilde{\mathfrak{S}}^{(q')}_{w_0w}(y)=\langle\tilde{\mathfrak{S}}^{(q)}_{w_0}(x,z),\tilde{\mathfrak{S}}^{(q')}_{w_0}(y,z)\rangle^{(z)},$$

where the upper index z means that the scalar product is taken in the z variables.

Corollary 2.

$$\sum_{w \in S_n} \Omega_w \Omega_w^* = C^{(q,q)}(x,x).$$

One can show that

$$C^{(q,q)}(x,x) = \sum_{w \in S_n} \tilde{\mathfrak{S}}_w(x) \tilde{\mathfrak{S}}_{w_0 w}(x) \equiv \det\left(\frac{\partial e_i(x|q)}{\partial x_j}\right)_{1 \leqslant i,j \leqslant n} \pmod{\tilde{I}}$$

Let us summarize our results. It follows from Theorem 2 that polynomials $\tilde{\mathfrak{S}}_w(x,0)$ are orthogonal with respect to the quantum pairing $\langle , \rangle_Q$. It is also clear that $\tilde{\mathfrak{S}}_w(x,0)|_{q=0} = \mathfrak{S}_w(x)$ and $\tilde{\mathfrak{S}}_w(x) = x^{c(w)}$+lower degree terms w.r.t. lexicographic order on the set of monomials. These two properties characterize the polynomials $\tilde{\mathfrak{S}}_w(x,0)$ uniquely; consequently, the polynomials $\tilde{\mathfrak{S}}_w(x,0)$ coincide with the quantum Schubert polynomials $\tilde{\mathfrak{S}}_w(x)$ from Definition 5. As a matter of fact, we obtain the Lascoux–Schützenberger type formula for quantum Schubert polynomials.

Theorem 4. *Let* $w \in S_n$. *Then*

$$\tilde{\mathfrak{S}}_w(x) = \partial_{ww_0}^{(y)} \tilde{\mathfrak{S}}_{w_0}(x,y)|_{y=0}.$$

The study of quantum Schubert polynomials was initiated in [7,17]. The Grassmannian case was considered earlier, in [1,4,27].

5. Quantization

5.1. Definition

Let $f \in P_n = \mathbf{Z}[x_1,\ldots,x_n]$ be a polynomial. According to the Interpolation formula

$$f(x) = \sum_{w \in S^{(n)}} \partial_w^{(y)} f(y) \mathfrak{S}_w(x,y).$$

We define a quantization $\tilde{f}$ of the function f by the rule

$$\tilde{f}(x) = \sum_{w \in S^{(n)}} \partial_w^{(y)} f(y) \tilde{\mathfrak{S}}_w(x,y)|_{\bar{P}_n}, \tag{15}$$

where for a polynomial $f \in \bar{P}_\infty$, the symbol $f|_{\bar{P}_m}$ means the restriction of f to the ring of polynomials $\bar{P}_m$, i.e. the specialization $x_{m+1} = x_{m+2} = \cdots = 0$ and $q_m = q_{m+1} = \cdots = 0$.

Hence, the quantization is a $\mathbf{Z}[q_1,\ldots,q_{n-1}]$-linear map $P_n \to \bar{P}_n$.

The main property of quantization is that it preserves the pairings, i.e.

$$\langle \tilde{f}, \tilde{g} \rangle_Q = \langle f, g \rangle, \quad f, g \in P_n. \tag{16}$$

It follows from (16) that the quantization map maps the ideal $I_n \subset \bar{P}_n$ into ideal $\tilde{I}_n \subset \bar{P}_n$.

Remark 5. (i) Quantization does not preserve multiplication, i.e. in general $\tilde{f}\cdot\tilde{g}\neq\widetilde{fg}$. For example, if $f=\sum_{i=1}^{n}\alpha_i x_i$ is a linear form, then (quantum Monk's formula, see [7] and our Section 9)

$$\tilde{f}\tilde{\mathfrak{S}}_w-\widetilde{f\mathfrak{S}_w}=\sum(\lambda_i-\lambda_j)q_{ij}\tilde{\mathfrak{S}}_{wt_{ij}},$$

summed over $i<j$ such that $l(w)=l(wt_{ij})+l(t_{ij})$. Here $q_{ij}=q_iq_{i+1}\ldots q_{j-1}$.

(ii) It is clear that if $f\in H_n$, then $\tilde{f}\in\bar{H}_n=H_n\otimes\mathbf{Z}[q_1,\ldots,q_{n-1}]$.

(iii) It follows from Proposition 9, that the quantum double Schubert polynomials $\tilde{\mathfrak{S}}_w(x,y)$ are the quantization of classical ones.

(iv) It follows from Interpolation formula and quantization procedure, that

- Quantum Schubert polynomials $\tilde{\mathfrak{S}}_w(x)$, $w\in S_n$ form a $\bar{I}$-basis in $\bar{P}_n$.
- Quantum Schubert polynomials $\tilde{\mathfrak{S}}_w(x)$, $w\in S^{(n)}$ form a $\mathbf{Z}[q_1,\ldots,q_{n-1}]$-basis of $\bar{P}_n$.
- Quantum Schubert polynomials $\tilde{\mathfrak{S}}_w(x)$, $w\in S_n$ form a $\mathbf{Z}[q_1,\ldots,q_{n-1}]$-basis of $\bar{H}_n$.

The proof of the statement (iv) can be found in [7].

Now we are going to describe another family of polynomials having a good quantization.

5.2. Elementary polynomials

Let $\delta:=\delta_n=(n-1,n-2,\ldots,1,0)$, and consider the set $\mathfrak{T}$ of sequences $I=(i_1,\ldots,i_n)\in\mathbf{Z}^n$ such that $0\leqslant i_j\leqslant n-j$ for all $j=1,\ldots,n$. It is clear that $|\mathfrak{T}|=n!$, and

- P_n is a free Λ_n-module of rank $n!$ with basis $\{x^I=x_1^{i_1}x_2^{i_2}\ldots x_n^{i_n}\mid I\in\mathfrak{T}\}$. Following [20], for each $I\in\mathfrak{T}$ let us define the elementary polynomial $e_I(x)$ as the following product:

$$e_I(x)=\prod_{k=1}^{n-1}e_{i_k}(x_1,\ldots,x_{n-k}).$$

- (Lascoux–Schützenberger [20]) P_n is a free Λ_n-module of rank $n!$ with basis $\{e_I(x)\mid I\in\mathfrak{T}\}$.

Definition 6. For each sequence $I\in\mathfrak{T}$ the quantized elementary polynomial $\tilde{e}_I(x)$ is defined to be

$$\tilde{e}_I(x)=\prod_{k=1}^{n-1}\tilde{e}_{i_k}(x_1,\ldots,x_{n-k}),$$

where $\tilde{e}_k(x_1,\ldots,x_m):=e_k(x_1,\ldots,x_m\mid q_1,\ldots,q_{m-1})$ are the quantum elementary polynomials.

Theorem 5. *Assume that $I\subset\delta$. Then $\tilde{e}_I(x)$ is the quantization of elementary polynomial $e_I(x)$.*

Proof. It is sufficient to prove the following. $\square$

Proposition 10. *If* $I \subset \delta$, *then*

$$\tilde{e}_I(x) = \sum_{w \in S_n} \tilde{\mathfrak{S}}_w(x)\eta(\partial_w e_I). \tag{17}$$

We will show that Proposition 10 follows from the quantum Cauchy formula. First of all, let us remark that

$$\tilde{\mathfrak{S}}_{w_0}(x, y) = \sum_{I \subset \delta} \tilde{e}_I(x) y^{\delta - I},$$

$$\tilde{\mathfrak{S}}_{w_0}(x, z) = \sum_{w \in S_n} \tilde{\mathfrak{S}}_{w_0 w w_0}(x) \mathfrak{S}_{w_0 w}(z).$$

Substituting these two expressions in (17), we obtain a formula for the classical Schubert polynomials:

$$\mathfrak{S}_w(y) = \sum_{I \subset \delta} \langle \tilde{e}_I(x), \tilde{\mathfrak{S}}_{w_0 w w_0}(x) \rangle_Q y^{\delta - I}. \tag{18}$$

It follows from (18) that

$$\langle \tilde{e}_I(x), \tilde{\mathfrak{S}}_{w_0 w w_0}(x) \rangle_Q = \langle e_I(x), \mathfrak{S}_{w_0 w w_0}(x) \rangle. \tag{19}$$

Conversely, the quantum Cauchy formula (12) follows from the classical one and (19). To continue, let us remark that

$$\begin{aligned}\langle e_I(x), \mathfrak{S}_{w_0 w w_0}(x) \rangle &= \langle e_I(x), \partial_{w_0 w^{-1}} \mathfrak{S}_{w_0}(x) \rangle \\ &= \langle \partial_{w w_0} e_I(x), \mathfrak{S}_{w_0}(x) \rangle = \eta(\partial_{w w_0} e_I(x)).\end{aligned}$$

As a corollary we obtain a formula for Schubert polynomials, which seems to be new:

$$\mathfrak{S}_w(x) = \sum_{I \subset \delta} \eta(\partial_{w w_0} e_I(x)) x^{\delta - I}. \tag{20}$$

Formula (20) gives a geometric interpretation of the coefficients $a_{I,w}$ of the Schubert polynomial $\mathfrak{S}_w(x) = \sum_{I \subset \delta} a_{I,w} x^{\delta - I}$ as the intersection numbers

$$a_{I,w} = \eta(\partial_{w w_0} e_I(x)) = \langle e_I(x), \mathfrak{S}_{w_0 w w_0}(x) \rangle \geqslant 0.$$

Finally, let us finish the proof of Theorem 5. We have

$$\begin{aligned}\sum_{I \subset \delta} \mathrm{RHS}(17) y^{\delta - I} &= \sum_{w \in S_n} \tilde{\mathfrak{S}}_w(x) \sum_{I \subset \delta} \eta(\partial_w e_I) y^{\delta - I} \\ &= \sum_{w \in S_n} \tilde{\mathfrak{S}}_w(x) \mathfrak{S}_{w w_0}(y) = \tilde{\mathfrak{S}}_{w_0}(x, y) = \sum_{I \subset \delta} \tilde{e}_I(x) y^{\delta - I}.\end{aligned}$$

Hence, RHS(17)$=\tilde{e}_I(x)$. $\square$

Corollary 3. *If* I *and* J *belong to* $\mathfrak{T}$, *then*

$$\langle \tilde{e}_I(x), \tilde{e}_J(x) \rangle_Q = \langle e_I(x), e_J(x) \rangle.$$

Remark 6. In the next section, we will give a proof of Corollary 3 using a geometric technique due to Ciocan–Fontanine [4] (see also [12]). Repeating our arguments in the reverse order, we see that the quantum Cauchy formula (12), as well as Theorems 1 and 3, follow directly from Corollary 3.

Using quantum Cauchy formula (12), we can describe a transition matrix between quantum Schubert polynomials and quantized elementary polynomials.

Theorem 6.

$$\tilde{\mathfrak{S}}_w(x) = \sum_{I \subset \delta} \tilde{e}_I(x)\eta(\partial_{ww_0} x^{\delta - I}).$$

Proof. It follows from Cauchy's formula that

$$\sum_{w \in S_n} \tilde{\mathfrak{S}}_w(x)\mathfrak{S}_{ww_0}(y) = \sum_{I \subset \delta} \tilde{e}_I(x) y^{\delta - I}.$$

Consequently,

$$\tilde{\mathfrak{S}}_{w_0 w w_0}(x) = \sum_{I \subset \delta} \tilde{e}_I(x)\langle y^{\delta - I}, \mathfrak{S}_w(y)\rangle.$$

Now we have

$$\begin{aligned}\langle y^{\delta-I}, \mathfrak{S}_w(y)\rangle &= \langle y^{\delta-I}, \partial_{w^{-1}w_0}\mathfrak{S}_{w_0}(y)\rangle = \langle \partial_{w_0 w} y^{\delta-I}, \mathfrak{S}_{w_0}(y)\rangle \\ &= \eta(\partial_{w_0 w} y^{\delta - I}). \qquad \square\end{aligned}$$

Example. Take the permutation $w = 24531 \in S_5$. It is easy to check that $ww_0 = 42135 = s_1 s_2 s_1 s_3$ and there exists 6 monomials x^I such that $I \subset (43210)$ and $\eta(\partial_{1213} x^I) \neq 0$. They are

$$\begin{array}{rcccccc} x^I: & x_1^2 x_2 x_3, & x_1^2 x_2 x_4, & x_1^2 x_3^2, & x_1 x_2^2 x_3, & x_1 x_2^2 x_4, & x_2^2 x_3^2 \\ \eta(\partial_{1213} x^I): & +1 & -1 & -1 & -1 & +1 & +1 \end{array}$$

We can check using Theorem 4, that

$$\tilde{\mathfrak{S}}_{24531}(x) = \tilde{e}_{2211}(x) - \tilde{e}_{2220}(x) - \tilde{e}_{2301}(x) - \tilde{e}_{3111}(x) + \tilde{e}_{3120}(x) + \tilde{e}_{4101}(x).$$

Now, let us consider a problem of how to quantize monomials.

Proposition 11. *Let $I \in \mathfrak{I}$, then*

$$x^I = \sum_{w \in S_n,\ l(w) = |I|} \eta(\partial_w x^I)\mathfrak{S}_w(x), \qquad \tilde{x}^I = \sum_{w \in S_n,\ l(w) = |I|} \eta(\partial_w x^I)\tilde{\mathfrak{S}}_w(x).$$

Corollary 4. *Let $v, w \in S_n$, then*

$$\sum_{I \subset \delta} \eta(\partial_v x^I)\eta(\partial_{ww_0} e_{\delta - I}(x)) = \delta_{v,w}.$$

Corollary 5.

$$\tilde{\mathfrak{S}}_w(x) = \sum_{I \subset \delta} \eta(\partial_{ww_0} e_{\delta-I}(x)) \tilde{x}^I.$$

Corollary 6.

$$\tilde{\mathfrak{S}}_{w_0}(x,y) = \sum_{I \subset \delta} \tilde{x}^I e_{\delta-I}(y).$$

Remark 7. (i) More generally, as proved in [15], the quantization of the flagged Schur function (see [21, Eqs. (3.1), (4.9) and (6.16)])

$$s_{\lambda/\mu}(X_1,\dots,X_n) = \det(h_{\lambda_i-\mu_j-i+j}(X_i))_{1 \leqslant i,j \leqslant n}$$

is given by

$$\tilde{s}_{\lambda/\mu}(X_1,\dots,X_n) = \det(\tilde{h}_{\lambda_i-\mu_j-i+j}(X_i))_{1 \leqslant i,j \leqslant n},$$

where $\tilde{h}_k(X)$ is the quantization of complete homogeneous symmetric function of degree k, and $X_1 \subset \cdots \subset X_n$ are the flagged sets of variables.

(ii) A determinantal formula for the quantum monomials $\tilde{x}^\lambda$ [15] as well as a quantum analog of the Billey–Jockusch–Stanley formula for Schubert polynomials in terms of compatible sequences [3] were obtained in [15].

(iii) (Generalized Cauchy's identity) A determinantal formula for the sum $\sum_{w \in S_n} \tilde{\mathfrak{S}}_w^{(q)}(x) \tilde{\mathfrak{S}}_{ww_0}^{(q)}(y)$ was obtained in [16].

Remark 8. To our knowledge, originally, construction of the quantization map, using a remarkable family of commuting operators X_i, appeared in [7]. We use a different definition of quantization map, but it can be shown that these two forms of quantization are equivalent. For original proofs of Theorem 5, and Proposition 15, see Corollary 4.6, Corollary 7.16 and Proposition 7.13 in [7].

6. Quantum cohomology ring of flag variety

Quantum cohomology ring of the flag variety Fl_n is a deformed ring of the ordinary cohomology ring $H^*(\mathrm{Fl}_n, \mathbf{Z})$. The structure constants of the quantum cohomology ring are given by the Gromov–Witten invariants. Let $\Omega_{w_1},\dots,\Omega_{w_m}$ $(w_i \in S_n)$ be Schubert cycles. We denote by $M_{\bar{d}}(\boldsymbol{P}^1, \mathrm{Fl}_n)$ the moduli space of morphisms from $\boldsymbol{P}^1$ to Fl_n of multidegree $\bar{d} = (d_1,\dots,d_{n-1})$. We consider the restriction of the universal map for $t \in \boldsymbol{P}^1$:

$$\mathrm{ev}_t : M_{\bar{d}}(\boldsymbol{P}^1, \mathrm{Fl}_n) \times \{t\} \hookrightarrow M_{\bar{d}}(\boldsymbol{P}^1, \mathrm{Fl}_n) \times \boldsymbol{P}^1 \xrightarrow{\mathrm{ev}} \mathrm{Fl}_n, \quad (f,p) \mapsto f(p).$$

Let $\Omega_w(t) = \mathrm{ev}_t^{-1}(\Omega_w)$.

Theorem 7 (Ciocan–Fontanine [5]). *If* $\sum_{i=1}^{m} l(w_i) = n(n-1)/2 + 2\sum d_i$ *and* $t_1,\dots,t_m \in \boldsymbol{P}^1$ *are distinct, then for general translates of* Ω_{w_i}, *the number of points in* $\bigcap_{i=1}^{m} \Omega_{w_i}(t_i)$ *is finite and independent of* t_i *and the translates of* Ω_{w_i}.

Definition 7. The Gromov–Witten invariant is defined as an intersection number

$$\langle \Omega_{w_1} \dots \Omega_{w_m} \rangle_{\bar d} = \begin{cases} \#\bigcap_i \Omega_{w_i}(t_i) & \text{if } \sum l(w_i) = n(n-1)/2 + 2\sum d_i, \\ 0 & \text{otherwise.} \end{cases}$$

Now we can define the quantum multiplication as a linear map

$$m_q\colon \mathrm{Sym}(H^*(\mathrm{Fl}_n, \mathbf{Z})[q_1,\dots,q_{n-1}]) \to H^*(\mathrm{Fl}_n, \mathbf{Z})[q_1,\dots,q_{n-1}]$$

given by

$$m_q\left(\prod_{i=1}^{m} \Omega_{w_i}\right) = \sum_{\bar d} q^{\bar d} \sum_{w} \langle \Omega_w \Omega_{w_1} \cdots \Omega_{w_m} \rangle_{\bar d} \Omega_w^*,$$

where $q^{\bar d} = q_1^{d_1} \cdots q_{n-1}^{d_{n-1}}$ and (Ω_w^*) is the dual basis of (Ω_w).

Then the quantum cohomology ring $QH^*(\mathrm{Fl}_n)$ is a commutative and associative $\mathbf{Z}[q_1,\dots,q_{n-1}]$-algebra.

Let $0 = E_0 \subset E_1 \subset \dots \subset E_n = \mathbf{C}^n \otimes \mathcal{O}_F$ be the universal flag of subbundles on Fl_n.

Theorem 8 (Givental and Kim [10], Ciocan–Fontanine [5]). *The small quantum cohomology ring is generated by* $x_i = c_1(E_{n-i+1}/E_{n-i})$, $i = 1,\dots,n$, *as a* $\mathbf{Z}[q_1,\dots,q_{n-1}]$-*algebra and*

$$QH^*(\mathrm{Fl}_n) \cong \mathbf{Z}[x_1,\dots,x_n,q_1,\dots,q_{n-1}]/(e_1(x|q),\dots,e_n(x|q)),$$

where $e_i(x|q)$ *is given by* (9)

It follows from Theorem 8 that any Schubert cycle Ω_w may be expressed as a polynomial $\hat{\mathfrak{S}}_w(x,q)$ in $QH^*(\mathrm{Fl}_n)$. The polynomial $\hat{\mathfrak{S}}_w(x,q)$ is a deformation of the Schubert polynomial $\mathfrak{S}_w(x)$ and $\hat{\mathfrak{S}}_w(x,0) = \mathfrak{S}_w(x)$. Consider the correlation function

$$\langle \Omega_{w_1} \dots \Omega_{w_m} \rangle = \sum_{\bar d} q^{\bar d} \langle \Omega_{w_1} \dots \Omega_{w_m} \rangle_{\bar d}.$$

Then $\hat{\mathfrak{S}}_w(x;q)$ is characterized by the condition

$$\langle \Omega_w \Omega_{w_1} \dots \Omega_{w_m} \rangle = \langle \hat{\mathfrak{S}}_w(x;q) \Omega_{w_1} \dots \Omega_{w_m} \rangle$$

for any $w_1,\dots,w_m \in S_n$. $\hat{\mathfrak{S}}_w(x;q)$ is called a geometric quantum Schubert polynomial. By definition $\hat{\mathfrak{S}}_w(x;q) \in QH^{2l(w)}(\mathrm{Fl}_n)$.

7. Proofs of Theorem 3 and quantum Cauchy formula

Theorem 9. *Let $I\in\mathfrak{T}$. Then*

$$\langle \tilde{e}_I(x), \tilde{\mathfrak{S}}_w(x)\rangle_Q = \langle e_I(x), \mathfrak{S}_w(x)\rangle \quad \textit{for any permutation } w\in S_n.$$

Proof. The proof is based on the arguments due to Ciocan-Fontanine [4]; see also [5] for more detailed exposition. To begin with, let us recall his results. We consider the hyper-quot scheme $\mathscr{H}\mathscr{Q}_{\bar d}(\boldsymbol{P}^1,\mathrm{Fl}_n)$ associated with $\boldsymbol{P}^1$ with multidegree $\bar d=(d_1,\dots,d_{n-1})$. Let

$$\boldsymbol{C}^n\otimes\mathcal{O}\to\mathcal{T}_{n-1}\to\cdots\to\mathcal{T}_2\to\mathcal{T}_1\to 0$$

be the universal sequence of quotients on $\boldsymbol{P}^1\times\mathscr{H}\mathscr{Q}_{\bar d}(\boldsymbol{P}^1,\mathrm{Fl}_n)$ and

$$\mathscr{S}_i=\mathrm{Ker}\{\boldsymbol{C}^n\otimes\mathcal{O}\to\mathcal{T}_{n-i}\}.$$

We also consider the dual sequence $\boldsymbol{C}^n\otimes\mathcal{O}\to\mathscr{S}^*_{n-1}\to\cdots\to\mathscr{S}^*_1$. We fix a flag $0=V_0\subset V_1\subset\cdots\subset V_n=\boldsymbol{C}^n$ and define the subschema $D_w^{p,q}$ of $\boldsymbol{P}^1\times\mathscr{H}\mathscr{Q}_{\bar d}(\boldsymbol{P}^1,\mathrm{Fl}_n)$ as the locus where $\mathrm{rank}(V_p\otimes\mathcal{O}\to\mathscr{S}^*_q)\leqslant r_w(q,p)$, and $r_w(q,p):=\sharp\{i\mid i\leqslant q, w_i\leqslant p\}$. Let

$$D_w^{p,q}(t)=D_w^{p,q}\cap\{\{t\}\times\mathscr{H}\mathscr{Q}_{\bar d}(\boldsymbol{P}^1,\mathrm{Fl}_n)\}$$

$$\bar\Omega_w(t)=\bigcap_{p,q=1}^{n-1}D_w^{p,q}(t).$$

Then the class of $\bar\Omega_w(t)$ in the Chow ring $\mathrm{CH}^{l(w)}(\mathscr{H}\mathscr{Q}_{\bar d}(\boldsymbol{P}^1,\mathrm{Fl}_n))$ is independent of $t\in\boldsymbol{P}^1$ and the flag $V_0\subset\cdots\subset V_n$.

The boundary $\mathscr{H}\mathscr{Q}_{\bar d}(\boldsymbol{P}^1,\mathrm{Fl}_n)\backslash M_{\bar d}(\boldsymbol{P}^1,\mathrm{Fl}_n)$ consists of $n-1$ divisors $\boldsymbol{D}_1,\dots,\boldsymbol{D}_{n-1}$, which are birational, respectively, to

$$\boldsymbol{P}^1\times\mathscr{H}\mathscr{Q}_{\bar d_1}(\boldsymbol{P}^1,\mathrm{Fl}_n),\dots,\boldsymbol{P}^1\times\mathscr{H}\mathscr{Q}_{\bar d_{n-1}}(\boldsymbol{P}^1,\mathrm{Fl}_n),$$

where $\bar d_i=(d_1,\dots,d_i-1,\dots,d_{n-1})$. Let $x_i(t)=\bar\Omega_{s_i}(t)-\bar\Omega_{s_{i-1}}(t)$, then for any permutation $w\in S_n$ there exists an element $G_w(t)\in\mathrm{CH}_*(\bigcup_{i=1}^{n-2}\boldsymbol{D}_i)$ such that

$$\bar\Omega_{w^{-1}}(t)-\mathfrak{S}_w(x_1(t),\dots,x_{n-1}(t))=j_*(G_{w^{-1}}(t)),$$

where $j:\bigcup_{i=1}^{n-2}\boldsymbol{D}_i\to\mathscr{H}\mathscr{Q}_{\bar d}(\boldsymbol{P}^1,\mathrm{Fl}_n)$ is the inclusion. Let $[m,k]\in S_n$ be the permutation

$$\begin{pmatrix}1&2&\dots&m-k-1&m-k&m-k+1&\dots&m&m+1&\dots&n\\1&2&\dots&m-k-1&m&m-k&\dots&m-1&m+1&\dots&n\end{pmatrix}.$$

Then the geometric Schubert polynomial $\tilde{\mathfrak{S}}_{[m,k]^{-1}}(x)$ is the elementary symmetric function in $x_1,\dots,x_{m-1}$ of degree k. Let $I=(i_1,\dots,i_n)$. We have to calculate in Chow' ring $\mathrm{CH}^*(\mathscr{H}\mathscr{Q}_{\bar d}(\boldsymbol{P}^1,\mathrm{Fl}_n))$ the following object

$$\left(\overline{\bigcap_{v=1}^{n-1}\Omega_{[n-v+1,i_v]^{-1}}}\right)(t)\cdot\prod_{j=1}^{N}\bar\Omega_{w_j}(t_j)-\prod_{v=1}^{n-1}\mathfrak{S}_{[n-v+1,i_v]}(x(t))\cdot\prod_{j=1}^{N}\bar\Omega_{w_j}(t_j),\qquad(21)$$

for distinct $t, t_1, \dots, t_N$ and $w_1, \dots, w_N \in S_n$ such that

$$\sum_{v=1}^{n-1} i_v + \sum_{j=1}^{N} l(w_j) = n(n-1)/2 + 2\sum_{k=1}^{n-1} d_k,$$

where $\cap$ is the classical intersection product and

$$\left(\overline{\bigcap_{v=1}^{n-1} \Omega_{[n-v+1,i_v]^{-1}}}\right)(t)$$

is the corresponding degeneracy locus on $\mathscr{HQ}_{\bar{d}}(\boldsymbol{P}^1, \mathrm{Fl}_n)$. In order to calculate the expression (21), first we will prove that

$$\left(\overline{\bigcap_{v=1}^{n-1} \Omega_{[n-v+1,i_v]^{-1}}}\right)(t) \cdot \prod_{j=1}^{N} \bar{\Omega}_{w_j}(t_j) - \prod_{v=1}^{n-1} \bar{\Omega}_{[n-v+1,i_v]^{-1}}(t) \cdot \prod_{j=1}^{N} \bar{\Omega}_{w_j}(t_j) = 0.$$

The LHS of the last expression can be computed as the number of points in

$$\prod_{v=1}^{n-1} \bar{\Omega}_{[n-v+1,i_v]^{-1}}(t) \cdot \prod_{j=1}^{N} \bar{\Omega}_{w_j}(t_j)$$

supported on $\bigcup_{i=1}^{n-2} \boldsymbol{D}_i$. Let $j_{\bar{d}_k}$ be the natural rational map

$$\boldsymbol{P}^1 \times \mathscr{HQ}_{\bar{d}_k}(\boldsymbol{P}^1, \mathrm{Fl}_n) - \to \mathscr{HQ}_{\bar{d}}(\boldsymbol{P}^1, \mathrm{Fl}_n).$$

From Remark 3 in [4],

$$j_{\bar{d}_k}^{-1}(\bar{\Omega}_{[n-v+1,i_v]^{-1}}(t)) = \begin{cases} \boldsymbol{P}^1 \times \bar{\Omega}_{[n-v+1,i_v]^{-1}}(t) \cup \displaystyle\bigcap_{n-v-i_v+1 \leqslant p \leqslant n-v-1} D^{p,n-v-1}_{[n-v+1,i_v]^{-1}}(t) & \text{if } k = n-v, \\ \boldsymbol{P}^1 \times \bar{\Omega}_{[n-v+1,i_v]^{-1}}(t) & \text{otherwise,} \end{cases}$$

where

$$\bigcap_{n-v-i_v+1 \leqslant p \leqslant n-v-1} D^{p,n-v-1}_{[n-v+1,i_v]^{-1}}(t) = \{t\} \times \bar{\Omega}_{[n-v,i_v-1]^{-1}}(t).$$

Because, by assumption,

$$\sum_{v=1}^{n-1} i_v + \sum_{v=1}^{n-1} l(w_j) = \frac{n(n-1)}{2} + 2\sum_{k=1}^{n-1} d_k,$$

we have

$$\bar{\Omega}_{[k,i_{n-k}-1]^{-1}}(t) \cdot \prod_{v \neq n-k} \bar{\Omega}_{[n-v+1,i_v]^{-1}}(t) \cdot \prod_{j=1}^{N} \bar{\Omega}_{w_j}(t_j) = 0$$

on the hyper-quot scheme $\mathscr{HQ}_{\bar{d}_k}(\boldsymbol{P}^1, \mathrm{Fl}_n)$. Hence, we have equality:

$$\left(\overline{\bigcap_{v=1}^{n-1} \Omega_{[n-v+1,i_v]^{-1}}}\right)(t) \cdot \prod_{j=1}^{N} \bar{\Omega}_{w_j}(t_j) = \prod_{v=1}^{n-1} \bar{\Omega}_{[n-v+1,i_v]^{-1}}(t) \cdot \prod_{j=1}^{N} \bar{\Omega}_{w_j}(t_j).$$

Our next observation is that the intersection number in the RHS of the last equality is equal to

$$\prod_{v=1}^{n-1} \bar{\Omega}_{[n-v+1,i_v]^{-1}}(s_v) \cdot \prod_{j=1}^{N} \bar{\Omega}_{w_j}(t_j),$$

where we can choose $s_1,\dots,s_{n-1}$, $t_1,\dots,t_N \in P^1$ to be the pairwise distinct points, since the class $[\Omega_w(t')]$ in the Chow ring $\mathrm{CH}^*(\mathscr{H}Q_{\bar{d}}(P^1,\mathrm{Fl}_n))$ does not depend on the choice of $t' \in P^1$.

Now, we to use the following identity:

$$\prod_{k=1}^{m} a_k - \prod_{k=1}^{m} b_k = \sum_{k=1}^{m} \prod_{j=1}^{k-1} b_j (a_k - b_k) \prod_{j=k+1}^{m} a_j.$$

Let us take in the last equality $m = n-1$

$$a_k := \bar{\Omega}_{[n-k+1,i_k]^{-1}}(s_k), \qquad b_k := \mathfrak{S}_{[n-k+1,i_k]}(x(s_k)).$$

Then we obtain the following equality:

$$\prod_{v=1}^{n-1} \bar{\Omega}_{[n-v+1,i_v]^{-1}}(s_v) \prod_{j=1}^{N} \bar{\Omega}_{w_j}(t_j) - \prod_{v=1}^{n-1} \mathfrak{S}_{[n-v+1,i_v]}(x(s_v)) \prod_{j=1}^{N} \bar{\Omega}_{w_j}(t_j)$$
$$= \left\{ \sum_{k=1}^{n-1} \prod_{j=1}^{k-1} \bar{\Omega}_{[n-j+1,i_j]^{-1}}(s_j) \cdot j_*(G_{[n,i_k]^{-1}}(t)) \prod_{j=k+1}^{n-1} \mathfrak{S}_{[n-j+1,i_j]}(x(s_j)) \right\} \cdot B,$$

where $B := \prod_{j=1}^{N} \bar{\Omega}_{w_j}(t_j)$.

The contributions from $j_*((G_{[n,i_k]^{-1}}(t))$, $1 \leqslant k \leqslant n-1$, can be computed by using the arguments in [4]. Indeed, as in the proof of Theorem 4 in [4], the intersection number $(1 \leqslant k \leqslant n-1)$

$$\prod_{l=1}^{k-1} \bar{\Omega}_{[n-l+1,i_l]^{-1}}(s_l) \cdot j_*(G_{[n,i_k]^{-1}}(t)) \prod_{l=k+1}^{n-1} \mathfrak{S}_{[n-l+1,i_l]}(x(s_l)) \cdot B$$

is the number of points in

$$\prod_{l=1}^{k-1} \bar{\Omega}_{[n-l+1,i_l]^{-1}}(s_j) \prod_{l=k}^{n-1} \mathfrak{S}_{[n-l+1,i_l]}(x(s_l)) \cdot B$$

supported on $\bigcup_{j=1}^{n-2} D_i$. Hence, by induction, we have the following identity for correlation functions

$$\left\langle \left(\bigcap_{v=1}^{n-1} \Omega_{[n-v+1,i_v]^{-1}} \right) \cdot \prod_{j=1}^{N} \bar{\Omega}_{w_j} \right\rangle = \left\langle \prod_{v=1}^{n-1} \tilde{\mathfrak{S}}_{[n-v+1,i_v]} \cdot \prod_{j=1}^{N} \bar{\Omega}_{w_j} \right.$$

where $\cap$ is the classical intersection product and $\cdot$ is the product in

$$\mathrm{Sym}\, H^*(\mathrm{Fl}_n, \boldsymbol{Z})[q_1,\dots,q_{n-1}].$$

The last equality for correlation functions is equivalent to the following one:

$$m_q(\Omega_{[n,i_1]^{-1}} \cap \Omega_{[n-1,i_2]^{-1}}, \cap \cdots \cap \Omega_{[l,i_{n-1}]^{-1}}, *) = m_q(\tilde{e}_{i_1} \cdot \tilde{e}_{i_2} \cdots \tilde{e}_{i_{n-1}}, *).$$

This completes the proof. □

8. Correlation functions

8.1. Higher genus correlation function and the Vafa–Intriligator type formula

Fix a Riemann surface C of genus g. We denote by $M_d(C,F)$ the moduli space of morphism from C to Fl_n. One can define the higher genus Gromov–Witten invariants by a method which is similar to that in the case of genus zero [25].

We have the following recursion relation for higher genus correlation function corresponding to the generating function for higher genus Gromov–Witten invariants

$$\langle \Omega_{w_1} \ldots \Omega_{w_N} \rangle_g = \sum_{v \in S_n} \langle \Omega_{w_1} \ldots \Omega_{w_N} \Omega_v \Omega_v^* \rangle_{g-1}$$

(cf. [25]). From Corollary 2 and Theorem 11 we can deduce the Vafa–Intriligator type formula for higher genus correlation functions, namely, let $\langle P(x_1,\ldots,x_n)\rangle_g$ be the genus g correlation function corresponding to a polynomial P, then

$$\langle P(x_1,\ldots,x_n)\rangle_g = \mathrm{Res}_{\tilde{I}}(P\Phi^g) = \sum_{\tilde{e}_1=\cdots=\tilde{e}_n=0} P(x_1,\ldots,x_n)\det\left(\frac{\partial \tilde{e}_i}{\partial x_j}\right)^{-1} (\Phi(x_1,\ldots,x_n))^g,$$

where $\Phi(x) = \langle \tilde{\mathfrak{S}}_{w_0}(x,y), \tilde{\mathfrak{S}}_{w_0}(x,y)\rangle^{(y)} = \sum_{w \in S_n} \tilde{\mathfrak{S}}_w(x)\tilde{\mathfrak{S}}_{w_0 w}(x) = C^{(q,q)}(x,x)$.

To simplify the above formula, we use the following observation:

$$\Phi(x) := C^{(q,q)}(x,x) \equiv \det\left(\frac{\partial \tilde{e}_i}{\partial x_j}\right)_{1\leqslant i,j\leqslant n} \pmod{\tilde{I}}.$$

Thus, we obtain

Theorem 10 (Higher genus Vafa–Intriligator formula).

$$\langle P(x_1,\ldots,x_n)\rangle_g = \mathrm{Res}_{\tilde{I}}(P\Phi^g) = \sum_{\tilde{e}_1=\cdots=\tilde{e}_n=0} P(x_1,\ldots,x_n)\left(\det\left(\frac{\partial \tilde{e}_i}{\partial x_j}\right)\right)^{g-1},$$

where $\tilde{e}_i(z)$, $1\leqslant i\leqslant n$, are the quantum elementary polynomial of degree i in the variable $z=(z_1,\ldots,z_m)$, see Section 5.2.

Remark 9. The polynomial

$$C^{(q,q')}(x,y) := \sum_{w \in S_n} \hat{\mathfrak{S}}_w^{(q)}(x)\tilde{\mathfrak{S}}_{w_0 w}^{(q')}(y)$$

corresponds to the dual class of the diagonal in the quantum cohomology ring

$$QH^*(\mathrm{Fl}_n \times \mathrm{Fl}_n, (q,q')) = QH^*(\mathrm{Fl}_n, q) \otimes QH^*(\mathrm{Fl}_n, q').$$

8.2. Witten–Dijkgraaf–Verlinde–Verlinde equations for the symmetric group

The Witten–Dijkgraaf–Verlinde–Verlinde equations (WDVV-equations) are equations on the correlation functions $\langle \tilde{\mathfrak{S}}_u \tilde{\mathfrak{S}}_v \tilde{\mathfrak{S}}_w \rangle \in \boldsymbol{Z}[q_1, \ldots, q_{n-1}]$, where $u, v, w \in S_n$, see, e.g., [23], where one can find a definition of WDVV-equations and their connection with quantum cohomology. The correlation functions satisfy the following conditions:

(1) Normalization:

$$\langle 1 \tilde{\mathfrak{S}}_v \tilde{\mathfrak{S}}_w \rangle = \langle \mathfrak{S}_v, \mathfrak{S}_w \rangle.$$

(2) Initial data:

$$\langle \tilde{\mathfrak{S}}_{s_k} \tilde{\mathfrak{S}}_{s_k} \tilde{\mathfrak{S}}_{w_0} \rangle = q_k.$$

(3) Degree conditions:

$$\langle \tilde{\mathfrak{S}}_u \tilde{\mathfrak{S}}_v \tilde{\mathfrak{S}}_w \rangle = 0,$$

if either $l(u) + l(v) + l(w) < l(w_0)$, or difference $l(u) + l(v) + l(w) - l(w_0)$ is an odd positive integer.

(4) WDVV-equations:

$$\sum_v \langle \tilde{\mathfrak{S}}_{w_1} \tilde{\mathfrak{S}}_{w_2} \tilde{\mathfrak{S}}_v \rangle \langle \tilde{\mathfrak{S}}_{w_0 v} \tilde{\mathfrak{S}}_{w_3} \tilde{\mathfrak{S}}_{w_4} \rangle = \sum_v \langle \tilde{\mathfrak{S}}_{w_2} \tilde{\mathfrak{S}}_{w_3} \tilde{\mathfrak{S}}_v \rangle \langle \tilde{\mathfrak{S}}_{w_0 v} \tilde{\mathfrak{S}}_{w_1} \tilde{\mathfrak{S}}_{w_4} \rangle,$$

for any $w_1, w_2, w_3, w_4 \in S_n$.

Conjecture 1. Conditions (1)–(4) uniquely determine the correlation functions $\langle \tilde{\mathfrak{S}}_u \tilde{\mathfrak{S}}_v \tilde{\mathfrak{S}}_w \rangle$.

Remark 10. (1) Correlation function $\langle \tilde{\mathfrak{S}}_{w_1} \tilde{\mathfrak{S}}_{w_2} \tilde{\mathfrak{S}}_{w_3} \rangle$ is a generating function for the Gromov–Witten invariants:

$$\langle \tilde{\mathfrak{S}}_{w_1} \tilde{\mathfrak{S}}_{w_2} \tilde{\mathfrak{S}}_{w_3} \rangle := \sum_{\bar{d}} q^{\bar{d}} \langle \tilde{\mathfrak{S}}_{w_1} \tilde{\mathfrak{S}}_{w_2} \tilde{\mathfrak{S}}_{w_3} \rangle_{\bar{d}}.$$

(2) More generally,

$$\langle \tilde{\mathfrak{S}}_{s_k} \tilde{\mathfrak{S}}_{s_{ij}} \tilde{\mathfrak{S}}_{w_0} \rangle = \begin{cases} q_i \ldots q_{j-1} & \text{if } 1 \leqslant i \leqslant k < j \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

8.3. Residue formula

Theorem 11. *Correlation function $\langle P(x_1, \ldots, x_n) \rangle$ is given by the formula*

$$\langle P(x_1, \ldots, x_n) \rangle = \sum_{\tilde{e}_1(p) = \cdots = \tilde{e}_n(p) = 0} \operatorname{Res}_p \left(\frac{P(x_1, \ldots, x_n)\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n}{\tilde{e}_1 \cdots \tilde{e}_n} \right).$$

Proof. If the polynomial $P(x_1,\dots,x_n)$ is in the ideal generated by $\tilde{e}_1,\dots,\tilde{e}_n$, then the left and right hand sides of the formula are zero. Hence, it is sufficient to prove that

$$\sum \operatorname{Res}_p \left(\frac{x_1^{v_1}\cdots x_{n-1}^{v_{n-1}}\, \mathrm{d}x_1\wedge\cdots\wedge \mathrm{d}x_n}{\tilde{e}_1\cdots\tilde{e}_n} \right)$$
$$= \begin{cases} 1 & \text{if } (v_1,\dots,v_{n-1}) = (n-1,n-2,\dots,1), \\ 0 & \text{if } 0\leqslant v_1+\cdots+v_{n-1} < n(n-1)/2, 0\leqslant v_i\leqslant n-i. \end{cases}$$

We can extend the meromorphic form

$$\omega = \frac{x_1^{v_1}\cdots x_{n-1}^{v_{n-1}}\, \mathrm{d}x_1\wedge\cdots\wedge \mathrm{d}x_n}{\tilde{e}_1\cdots\tilde{e}_n}$$

on the affine space $\boldsymbol{A}^n_{(x_1,\dots,x_n)}$ to $(\boldsymbol{P}^1)^n$. For each subset $I\subset\{1,\dots,n\}$, we consider the coordinate chart $U_J=\boldsymbol{A}^n_{(z_1^J,\dots,z_n^J)}$, where

$$z_i^J = \begin{cases} x_i & \text{if } i\neq J, \\ 1/x_i & \text{otherwise.} \end{cases}$$

Then $(\boldsymbol{P}^1)^n=\bigcup_J U_J$. Let

$$\bar{e}_j^J(z_1^J,\dots,z_n^J) = \left(\prod_{i\in J} z_i^J\right) \tilde{e}_j(x_1,\dots,x_n).$$

Then

$$\omega = (-1)^{\sharp J} \frac{x_1^{v_1}\cdots x_{n-1}^{v_{n-1}}(\prod_{i\in J} z_i^J)^{n-2}\, \mathrm{d}z_1^J\wedge\cdots\wedge \mathrm{d}z_n^J}{\bar{e}_1^J\cdots\bar{e}_n^J}$$

on $U_\phi\cap U_J$. If $\sharp J=j$, then there exists a polynomial $Q_i(z_1^J,\dots,z_n^J)$ for $i\in J$ such that

$$\bar{e}_j^I(z_1^J,\dots,z_n^J) = 1+\sum_{i\in J} z_i^J Q_i.$$

This follows from

$$\tilde{e}_j(x_1,\dots,x_n) = e_j(x_1,\dots,x_n) + (\text{terms of lower degree}).$$

Therefore $\bar{e}_1^J,\dots\bar{e}_n^J$ do not have common zero on $B_J=\{(z_1^J,\dots,z_n^J)\in U_J \mid z_i^J=0, i\in J\}$.

From the residue theorem, if $0\leqslant v_1+\cdots+v_n < n(n-1)/2$, $0\leqslant v_i\leqslant n-i$, then

$$\sum \operatorname{Res}_p \left(\frac{x_1^{v_1}\cdots x_{n-1}^{v_{n-1}}\, \mathrm{d}x_1\wedge\cdots\wedge \mathrm{d}x_n}{\tilde{e}_1\cdots\tilde{e}_n} \right) = 0.$$

On the other hand,

$$\sum_{\tilde{e}_i(p)=0} \operatorname{Res}_p \left(\frac{x_1^{n-1}\cdots x_{n-1}\, \mathrm{d}x_1\wedge\cdots\wedge \mathrm{d}x_n}{\tilde{e}_1\cdots\tilde{e}_n} \right) = -\sum_p \operatorname{Res}_p\omega,$$

where p runs over the common zeros of $\bar{e}_1^J,\dots,\bar{e}_n^J$ in $\bigcup_{1\in J} B_J$. Let $y_1=1/x_1$, $z=(y_1,x_2,\dots,x_n)$ and $\bar{e}_1^*(z)=y_1(1+y_1(x_2+\cdots+x_n))$. Then by induction we have

$$-\sum_p \mathrm{Res}_p\omega = \sum_{\substack{\bar{e}_1^*=\bar{e}_2^{\{1\}}=\cdots=\bar{e}_n^{\{1\}}=0\\ \text{in the locus } \{y_1=0\}}} \mathrm{Res}_p\left(\frac{x_2^{n-2}\cdots x_{n-1}\,\mathrm{d}y_1\wedge\cdots\wedge\mathrm{d}x_n}{\bar{e}_1^*(z)\cdot\bar{e}_2^{\{1\}}(z)\cdots\bar{e}_N^{\{1\}}(z)}\right)$$

$$=\sum_p \mathrm{Res}_p\left(\frac{x_2^{n-2}\cdots x_{n-1}\,\mathrm{d}x_2\wedge\cdots\wedge\mathrm{d}x_n}{\tilde{e}_1(x_2,\dots,x_n)\cdots\tilde{e}_{n-1}(x_2,\dots,x_n)}\right)=1. \qquad \square$$

From the residue formula, the correlation function is given by the quantum residue $\mathrm{Res}_{\tilde{I}}$, namely,

$$\langle P(x_1,\dots,x_{n-1})\rangle = \mathrm{Res}_{\tilde{I}}P(t_1,\dots,t_{n-1}).$$

In order to relate the quantum residue with the classical one, we consider the quantum residue generating function

$$\Psi(t)=\left\langle \prod_{i=1}^{n-1}\frac{t_i}{t_i-x_i}\right\rangle = \sum_{v\in(\boldsymbol{Z}_{\geqslant 0})^{n-1}}\langle x^v\rangle t^{-v}.$$

Then, we have

$$\mathrm{Res}_{\tilde{I}}P(t_1,\dots,t_{n-1})=\mathrm{Res}_I(P(x_1,\dots,x_{n-1})\Psi(x)).$$

Hence, it is important to determine the generating function $\Psi(t)$. Let

$$f_i(t)=t^n+\sum_{j=0}^{n-1}\gamma_{n-j}^{(i)}t^j$$

be the characteristic polynomial of the quantum multiplication by x_i with respect to the basis consisting of the quantum Schubert polynomials. Let us consider the $(n!+1)\times(n!+1)$-matrix $C_n(t)$ such that

$$(C_n(t))_{1,j}=\frac{(-1)^{j-1}t^{n-j+2}}{(j-1)!};$$

$$(C_n(t))_{i,j}=\begin{cases}\dfrac{(-1)^n}{(j-1)!}\dbinom{i-2}{n-j+1}t^{j-2} & \text{if } i\geqslant 2,\ i+j\geqslant n+2,\\ 0 & \text{otherwise.}\end{cases}$$

We define the differential operator D_i by

$$D_i=(\gamma_{n!}^{(i)},\gamma_{n!-1}^{(i)},\dots,1)\cdot C_n(t_i)\cdot{}^t\,(1,\partial/\partial t_i,\dots,(\partial/\partial t_i)^{n!}).$$

Proposition 12. *The generating function $\Psi(t)$ satisfies the system of differential equations*

$$D_i\Psi(t)=0,\quad 1\leqslant i\leqslant n-1.$$

Conversely, these differential equations and the initial values $\langle x_1^{v_1}\cdots x_{n-1}^{v_{n-1}}\rangle$ for $0\leqslant v_i\leqslant n!-1$ determine the generating function uniquely.

Proof. Let x be a variable. Since

$$\begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ \vdots \\ x^{n!} \end{pmatrix} = C_n \cdot \begin{pmatrix} t(t-x)^{n!} \\ -x(t-x)^{n!-1} \\ 2!x(t-x)^{n!-2} \\ -3!x(t-x)^{n!-3} \\ \vdots \\ (-1)^{n!}(n!)!x \end{pmatrix},$$

we have

$$D_i \Psi(t) = \left\langle \frac{f_i(x_i)}{(t_i - x_i)^{n!+1}} \prod_{j \neq i} \frac{t_j}{t_j - x_j} \right\rangle = 0.$$

On the other hand, the recursive relations

$$\langle f_i(x_i) P(x_1, \dots, x_{n-1}) \rangle = 0$$

with the initial values $\langle x_1^{v_1} \cdots x_{n-1}^{v_{n-1}} \rangle$ for $0 \leqslant v_i \leqslant n! - 1$ determine the correlation function uniquely. $\square$

Remark 11. We can also consider another generating function $\langle \exp(x_1 t_1 + \cdots + x_n t_n) \rangle$. This is the generating volume function in [10]. This generating function satisfies

$$\tilde{e}_i \left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n} \right) \langle \exp(x_1 t_1 + \cdots + x_n t_n) \rangle = 0 \quad \text{for } 1 \leqslant i \leqslant n.$$

In the case of $n = 3$, we can calculate the generating function $\Psi(t)$ explicitly. To do this, we define the functions $g_v(t_1)$ and $h_v(t_2)$ by the formulae

$$\frac{t_1}{t_1 - x_1} = \sum_{v \in S_3} g_v(t_1) \mathscr{S}_v^q, \qquad \frac{t_2}{t_2 - x_2} = \sum_{v \in S_3} h_v(t_2) \mathscr{S}_v^q$$

in the quantum cohomology ring $QH^*(\mathrm{Fl}_3)$. Then, we have

$$\left\langle \frac{t_1}{t_1 - x_1} \frac{t_2}{t_2 - x_2} \right\rangle = \sum_{v \in S_3} g_v(t_1) h_{v w_0}(t_2).$$

Since $\langle x_1^5 \rangle = q_1$, $x_2 = q_1^{-1} x_1^3 - 2x_1$, and $QH^*(\mathrm{Fl}_3) \simeq \mathbf{Z}[q_1, q_2][x_1]/(f_1(x_1))$, the correlation function $\langle P(x_1, x_2) \rangle$ is expressed as

$$\langle P(x_1, x_2) \rangle = q_1 \operatorname{Res}_{f_1} P(x_1, q_1^{-1} x_1^3 - 2x_1) = q_1 \sum_{f_1(\mu)=0} \frac{1}{f_1'(\mu)} P(\mu, q_1^{-1} \mu^3 - 2\mu).$$

Similarly,

$$\langle P(x_1, x_2) \rangle = (q_2 - q_2) \sum_{f_2(\mu)=0} \frac{1}{f_2'(\mu)} P((q_1 - q_2)^{-1} (\mu^3 - (2q_1 + q_2)\mu), \mu) \text{ also holds.}$$

The functions g_v and h_v are given as follows:

$$f_1(t_1)g_{121}(t_1)=q_1t_1,$$

$$f_1(t_1)g_{12}(t_1)=q_1t_1^2,$$

$$f_1(t_1)g_{21}(t_1)=t_1^2(t_1^2-q_1),$$

$$f_1(t_1)g_2(t_1)=q_1t_1(t_1^2-q_1),$$

$$f_1(t_1)g_1(t_1)=t_1(t_1^4-q_1t_1^2+q_1^2),$$

$$f_1(t_1)g_{id}(t_1)=t_1^2(t_1^4-q_1t_1^2+q_1^2),$$

$$f_2(t_2)h_{121}(t_2)=(q_2-q_1)t_2,$$

$$f_2(t_2)h_{12}(t_2)=t_2^2(2q_1+q_2-t_2^2),$$

$$f_2(t_2)h_{21}(t_2)=t_2^2(q_1+2q_2-t_2^2),$$

$$f_2(t_2)h_2(t_2)=\frac{(q_1+q_2)t_2^5-q_1(q_1+q_2)t_2^3+q_1q_2(2q_2-q_1)t_2}{(q_1-q_2)},$$

$$f_2(t_2)h_1(t_2)=-t_2^5+(2q_1+q_2)t_2^3+q_1(q_2-q_1)t_2,$$

$$f_2(t_2)h_{id}(t_2)=\frac{(q_1+q_2)t_2^6-2q_1t_2^2+(q_1^3+2q_1^2q_2-q_2^3)t_2^2}{(q_1-q_2)}.$$

The characteristic polynomials f_1 and f_2 are given by

$$f_1(t)=(t^2-q_1)^3-q_1^2q_2,$$
$$f_2(t)=t^6-3(q_1+q_2)t^4+3(q_1^2+q_1q_2+q_2^2)t^2-q_1^3+q_1^2q_2+q_1q_2^2-q_2^3.$$

Hence, we have

$$\begin{aligned} D_1 &= \frac{1}{720t_1}f_1(t_1)\frac{\partial^6}{\partial t_1^6}+\left(\frac{q_1^3+q_1^2q_2}{120t_1^2}+\frac{1}{40}q_1^2-\frac{3}{40}q_1t_1^2+\frac{1}{24}t_1^4\right)\frac{\partial^5}{\partial t_1^5} \\ &\quad+\left(-\frac{q_1^3+q_1^2q_2}{24t_1^3}-\frac{3}{8}q_1t_1+\frac{5}{12}t_1^3\right)\frac{\partial^4}{\partial t_1^4}+\left(\frac{q_1^3+q_1^2q_2}{6t_1^4}-\frac{1}{2}q_1+\frac{5}{3}t_1^2\right)\frac{\partial^3}{\partial t_1^3} \\ &\quad+\left(-\frac{q_1^3+q_1^2q_2}{2t_1^5}+\frac{5}{2}t_1\right)\frac{\partial^2}{\partial t_1^2}+\left(\frac{q_1^3+q_1^2q_2}{t_1^6}+1\right)\frac{\partial}{\partial t_1}-\frac{q_1^3+q_1^2q_2}{t_1^7}, \\ D_2 &= \frac{f_2(t_2)}{720t_2}\frac{\partial^6}{\partial t_2^6}+\left(\frac{q_1^3-q_1^2q_2-q_1q_2^2+q_2^3}{120t_2^2}+\frac{q_1^2+q_1q_2+q_2^2}{40}\right. \\ &\quad\left.+\frac{3}{40}(q_1+q_2)t_2^2+\frac{t_2^4}{24}\right)\frac{\partial^5}{\partial t_2^5} \end{aligned}$$

$$+\left(-\frac{q_1^3-q_1^2q_2-q_1q_2^2+q_2^3}{24t_2^3}-\frac{3}{8}(q_1+q_2)t_2+\frac{5}{12}t_2^3\right)\frac{\partial^4}{\partial t_2^4}$$
$$+\left(\frac{q_1^3-q_1^2q_2-q_1q_2^2+q_2^3}{6t_2^4}-\frac{1}{2}(q_1+q_2)+\frac{5}{3}t_2^2\right)\frac{\partial^3}{\partial t_2^3}$$
$$+\left(-\frac{q_1^3-q_1^2q_2-q_1q_2^2+q_2^3}{2t_2^5}+\frac{5}{2}t_2\right)\frac{\partial^2}{\partial t_2^2}$$
$$+\left(\frac{q_1^3-q_1^2q_2-q_1q_2^2+q_2^3}{t_2^6}+1\right)\frac{\partial}{\partial t_2}-\frac{q_1^3-q_1^2q_2-q_1q_2^2+q_2^3}{t_2^7}.$$

9. Ehresmann–Bruhat graph and quantum Pieri rule

Let us recall that the Ehresmann–Bruhat order denoted by $\leqslant$, is the partial order on S_n that is the transitive closure of the relation $\to$. Relation $v\to w$ means that

(1) $l(w)=l(v)+1$,

(2) $w=v\cdot t$ where t is a transposition.

In other words, if v and w are permutations, $v\leqslant w$ means that there exists $r\geqslant 0$ and $v_0,v_1,\ldots,v_r$ in S_n such that $v=v_0\to v_1\to\cdots\to v_r=w$.

Now let us define the Ehresmann–Bruhat graph on S_n. First of all, we define a relation $v\leftarrow w$ (see, also, [7]). Relation $v\leftarrow w$, $v,w\in S_n$, means that

(1) $w=v\cdot t$, where t is a transposition,

(2) $l(w)\geqslant l(v)+l(v^{-1}w)$.

Remark 12. (i) As it follows from [21, Eq. (1.10), Condition 2] is equivalent to the following one: $(2')$ $w(i)<w(j)$ and for all k such that $i<k<j$ we have $w(i)<w(k)<w(j)$.

(ii) If $w=vt_{i,i+1}$ and $l(w)=l(v)+1$ (i.e. $v\to w$ in the Bruhat order), then we have also an arrow $v\leftarrow w$. This is clear because in our case we have $l(w)=l(v)+l(t_{i,i+1})$.

We define a weight of an arrow $v\leftarrow w$, denoted by $wt(v\leftarrow w)$, to be equal to the product $q_i\ldots q_{i+s-1}$, if $t=t_{ij}$ and $2s{:=}l(w)+1-l(v)$ (see, also, [7]). We assume that weight of any arrow $v\to w$ is equal to 1.

Let us say that an arrow $v\leftarrow w$ (resp. $v\to w$) has a color k if $w=vt_{ij}$ and $1\leqslant i\leqslant k<j\leqslant n$.

Definition 8. The Ehresmann–Bruhat graph on S_n is the set of all permutations $w\in S_n$ (vertices) together with the set of directed edges between pairs of permutations according to the relations $\leftarrow$ or $\to$.

The Ehresmann–Bruhat path (BE-path) between two (ordered) permutations v and w in S_n (notation $v\Longleftarrow w$) is a sequence of permutations (for some $r\geqslant 0$) $v_0,v_1,\ldots,v_r$

in S_n such that

$$v = v_0 \rightleftharpoons v_1 \rightleftharpoons v_2 \rightleftharpoons \cdots \rightleftharpoons v_r = w, \tag{22}$$

where symbol $v_i \rightleftharpoons v_{i+1}$ means either $v_i \longrightarrow v_{i+1}$ or $v_i \longleftarrow v_{i+1}$.

We denote the number r in a representation (22) by $l(v \Longleftarrow w)$.

Let us define a weight of a BE-path $v \Longleftarrow w$ as follows:

$$wt(v \mapsto w) = \prod_{i=0}^{r-1} wt(v_i \rightleftharpoons v_{i+1}).$$

We state that BE-path $v \Longleftarrow w$ has a color k, notation $v \stackrel{k}{\Longleftarrow} w$, if in the representation (22) all arrows $v_i \rightleftharpoons v_{i+1}$ $(i = 0, \ldots, r-1)$ have the same color k.

Theorem 12 (Quantum Pieri's rule). *Let us consider the Grassmannian permutation*

$$[b, d] = (1, 2, \ldots, b-d-1, b, b-d, b-d+1 \ldots, b-1, b+1, \ldots, n),$$

for $2 \leqslant b \leqslant n$, $1 \leqslant d \leqslant b$. *Then*

$$\tilde{\mathfrak{S}}_{[b,d]} \cdot \tilde{\mathfrak{S}}_v \equiv \sum wt(v \stackrel{b}{\Longleftarrow} w) \tilde{\mathfrak{S}}_w (\mathrm{mod}\, \tilde{I}_n),$$

where the sum runs over all BE-paths $v \stackrel{b}{\Longleftarrow} w$ *with color* b, *such that*

(1) $l(v \Longleftarrow w) = d$;

(2) *if* $v_l = v_{l+1}(i_l j_l)$ $(l = 0, \ldots, d-1)$, *then all* i_l *are different.*

(Note that $\mathfrak{S}_{[b,d]} = e_d(x_1, \ldots, x_{b-1})$).

Proof of Theorem 12 (sketch). It is sufficient to consider the case $d = 1$ (induction!). In this case, we use a quantum analog of Kohnert–Veigneau's method [18]. First, we prove the quantum Pieri rule (for $d = 1$) for double quantum Schubert polynomials and then take $y = 0$ (see Theorem 4).

Proposition 13 (Quantum Pieri's rule for $\tilde{\mathfrak{S}}_{w_0}(x, y)$).

$$(x_j + y_{n+1-j}) \tilde{\mathfrak{S}}_{w_0}(x, y) \equiv \sum_{i<j} q_{ij} \tilde{\mathfrak{S}}_{w_0 t_{ij}}(x, y) - \sum_{j<k} q_{jk} \tilde{\mathfrak{S}}_{w_0 t_{jk}}(x, y) \ (\mathrm{mod}\ \tilde{J}) \tag{23}$$

where $q_{ij} := q_i q_{i+1} \ldots q_{j-1}$, *if* $i < j$;

$\tilde{J}$ *is the ideal in the ring* $\mathbf{Z}[x_1, x_2, \ldots, x_n, y_1, \ldots, y_n, q_1, \ldots, q_{n-1}]$

generated by

$$e_i(x_1, \ldots x_n \mid q_1, \ldots, q_{n-1}) + (-1)^{i-1} e_i(y_1, \ldots, y_n), \quad 1 \leqslant i \leqslant n,$$

and $e_k(x_1, \ldots, x_n \mid q_1, \ldots, q_{n-1})$ *is the kth quantum elementary symmetric function.*

Applying the generalized Monk formula (see Section 2.2) to (23), we obtain

Corollary 7 (Equivariant quantum Pieri's rule).

$$
\begin{aligned}
& x_j \tilde{\mathfrak{S}}_w(x,y) + y_{w_j} \tilde{\mathfrak{S}}_w(x,y) \\
& \quad \equiv \sum_{j<k, l(wt_{jk})=l(w)+1} \tilde{\mathfrak{S}}_{wt_{jk}}(x,y) + \sum_{j<k, l(w)=l(wt_{jk})+l(t_{jk})} q_{jk} \tilde{\mathfrak{S}}_{wt_{jk}}(x,y) \\
& \qquad - \sum_{i<j, l(wt_{ij})=l(w)+1} \tilde{\mathfrak{S}}_{wt_{ij}}(x,y) - \sum_{i<j, l(w)=l(wt_{ij})+l(t_{ij})} q_{ij} \tilde{\mathfrak{S}}_{wt_{ij}}(x,y) \pmod{\tilde{J}}.
\end{aligned}
$$

Remark 13. (i) $\tilde{\mathfrak{S}}_{[b,d]} = e_d(x_1,\ldots,x_b \,|\, q_1,\ldots,q_{b-1})$ coincides with quantum elementary symmetric function.

(ii) It is clear that $\tilde{\mathfrak{S}}_{s_k}^2 = \tilde{\mathfrak{S}}_{s_{k+1}s_k} + \tilde{\mathfrak{S}}_{s_{k-1}s_k} + q_k$, i.e. $\langle \tilde{\mathfrak{S}}_k \tilde{\mathfrak{S}}_k \tilde{\mathfrak{S}}_{w_0} \rangle = q_k$.

(iii) To our knowledge, in the classical case $q = 0$, the Pieri rule for Schubert polynomials was first stated in [19, Eq. (2.2)]. Our formulation of Theorem 12 is very close to that given in [2]. The difference is: we use the paths in the Ehresmann–Bruhat graph (quantum case) instead of the paths in the ordinary Ehresmann–Bruhat order (classical case). One can find clear proof of Monk's formula in the Macdonald's book [21, Eq. (4.15)]. It is the proof that was generalized in [7] for the case of quantum Schubert polynomials. Recently, Sottile [26] gave a proof of the Pieri rule based on geometric approach, and Postnikov [24] gave an algebraic proof of the generalized quantum Pieri rule based on 'noncommutative' Schubert calculus developed by Fomin and Kirillov [8].

10. For further reading

The following reference is also of interest to the reader: [14].

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